

Universal Pairs of Rotations

Maris Ozols, Laura Mancinska,
Andrew Childs, Debbie Leung

Abstract

Assume you are given a sphere in three dimensions and a pair of rotations about two fixed axis by two fixed angles. Can you compose these rotations to approximate any given rotation to any desired accuracy?

We will characterize the cases when it is possible and show how one should compose the two rotations to approximate an arbitrary rotation.

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UNIVERSAL PAIRS OF ROTATIONS

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OF ROTATIONS

MARIS OZOLS

UNIVERSAL PAIRS OF ROTATIONS

MARIS OZOLS

LAURA MANCINSKA

UNIVERSAL PAIRS OF ROTATIONS

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ANDREW CHILDS

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UNIVERSAL PAIRS OF ROTATIONS

MARIS OZOLS

LAURA MANCINSKA

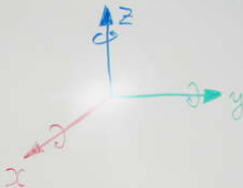
ANDREW CHILDS

DEBBIE LEUNG

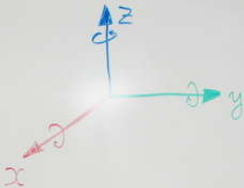
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ARE 3 AXES ENOUGH?

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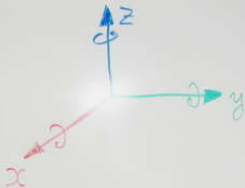


ARE 3 AXES ENOUGH?



ARE 2 AXES ENOUGH?

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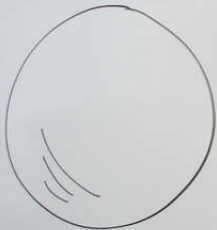


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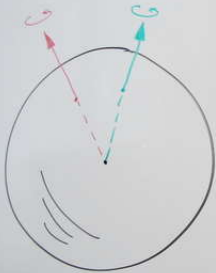
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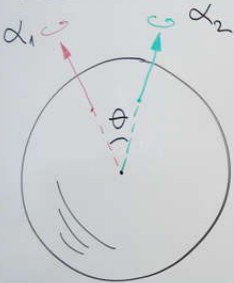
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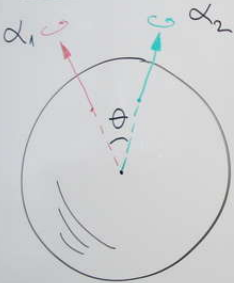
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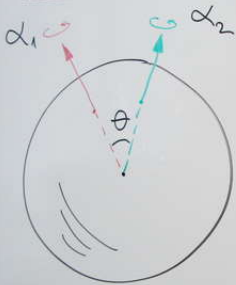


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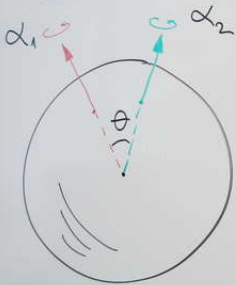
IS THE PAIR OF
ROTATIONS $(\theta, \alpha_1, \alpha_2)$
UNIVERSAL

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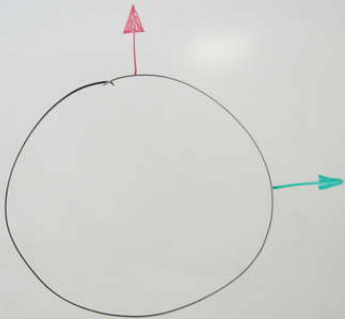
LET'S TRY...

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$(\theta, \alpha_1, \alpha_2)$

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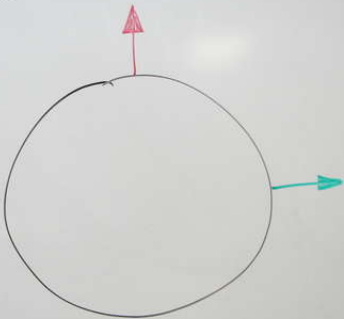
$(\theta, \alpha_1, \alpha_2)$



LET'S TRY ...

$$(\theta, \alpha_1, \alpha_2)$$

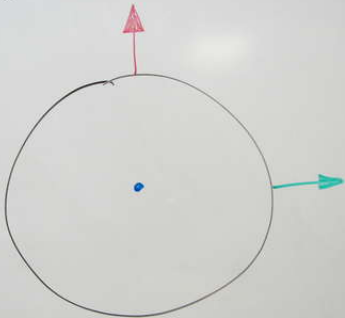
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LET'S TRY ...

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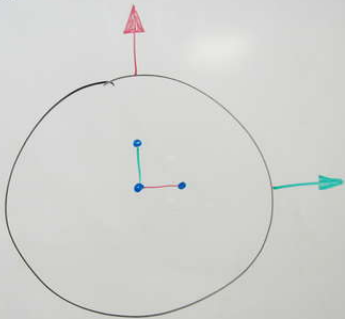
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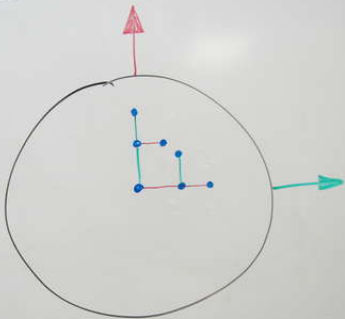
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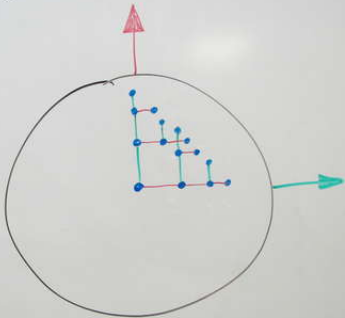
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LET'S TRY ...

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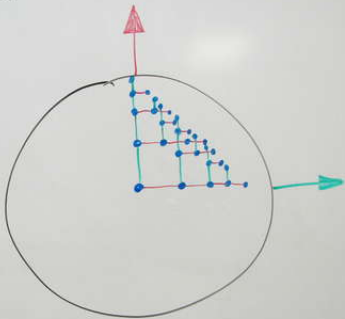
$$\equiv \frac{\pi}{2}$$



LET'S TRY ...

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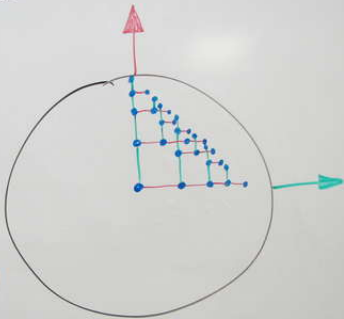


LET'S TRY ...

$$(\theta, \alpha_1, \alpha_2)$$

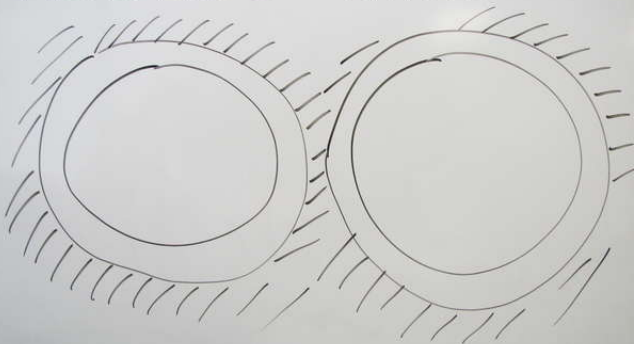
$$\parallel \frac{\pi}{2}$$

WHAT'S
GOING ON?

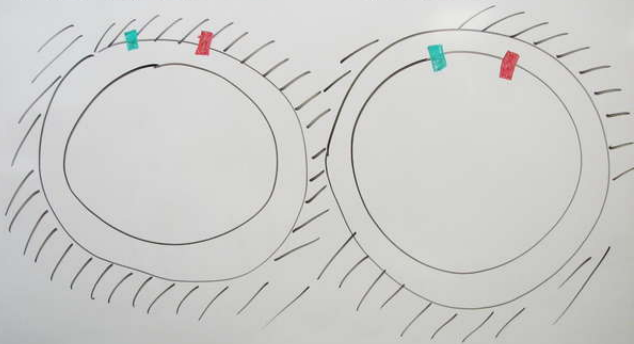


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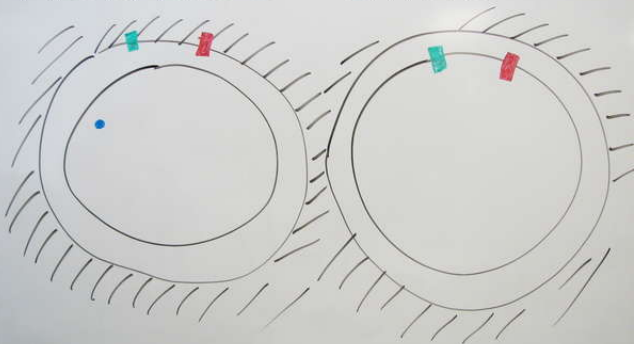
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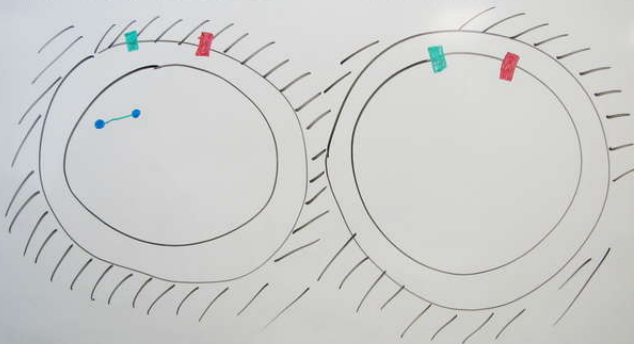
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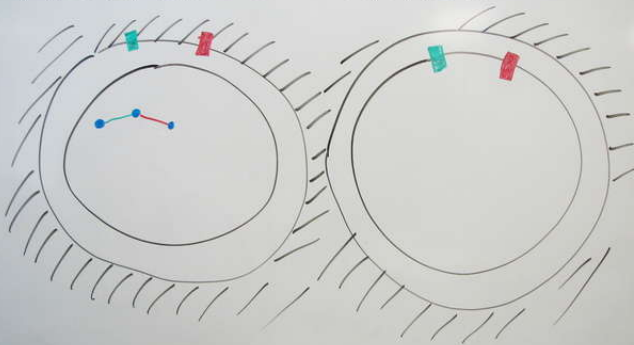
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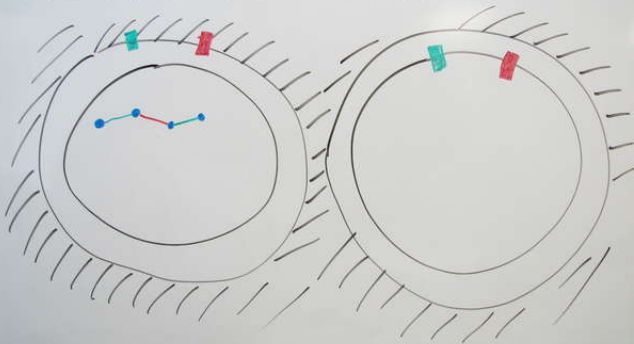
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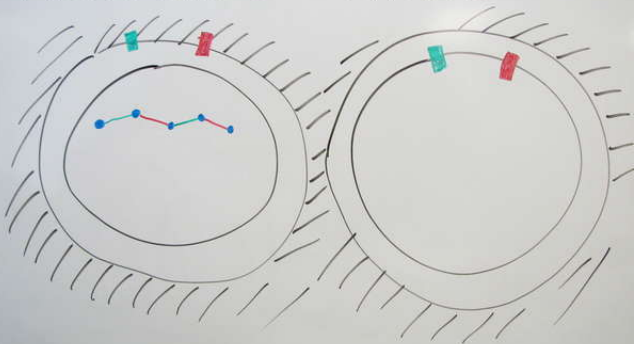
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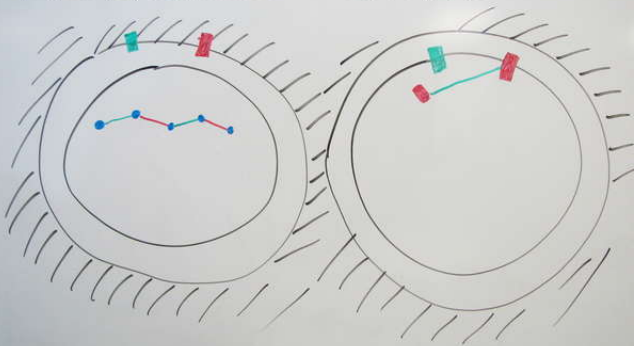
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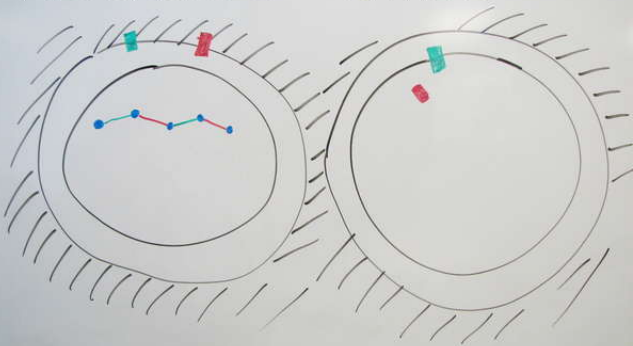
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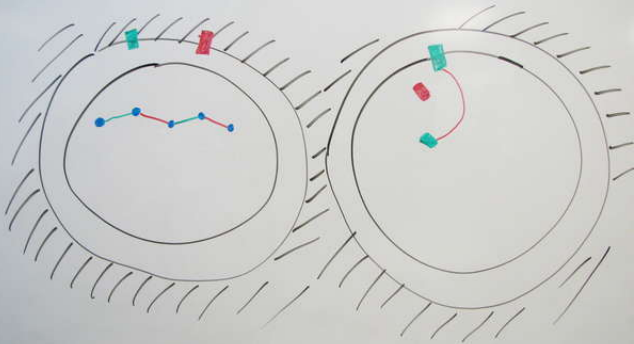
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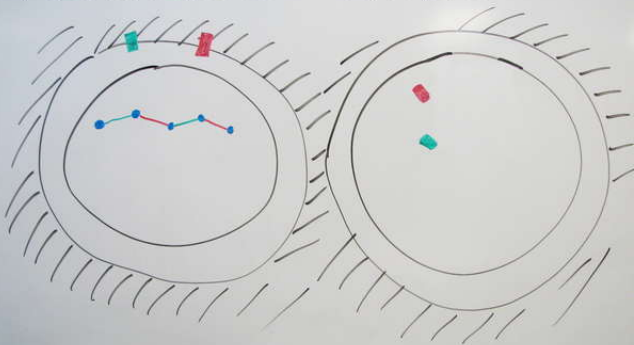
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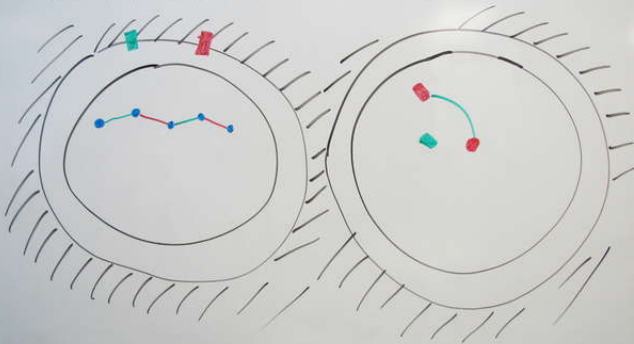
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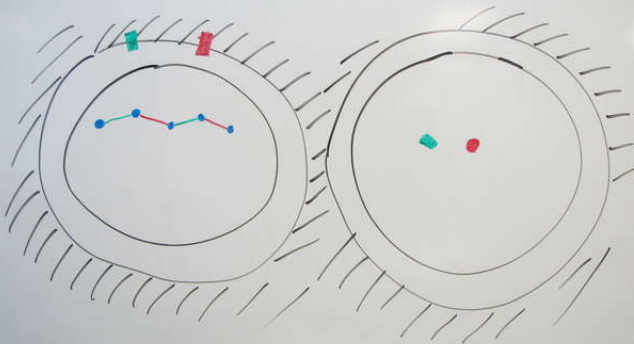
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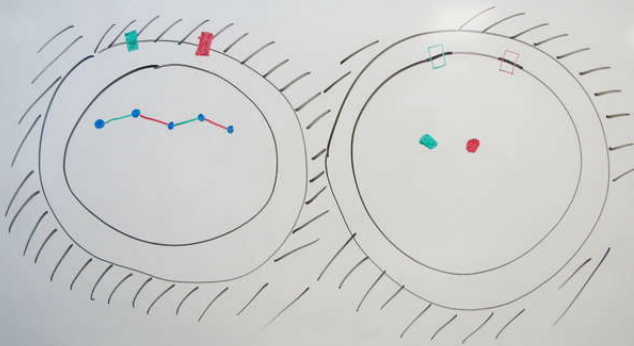
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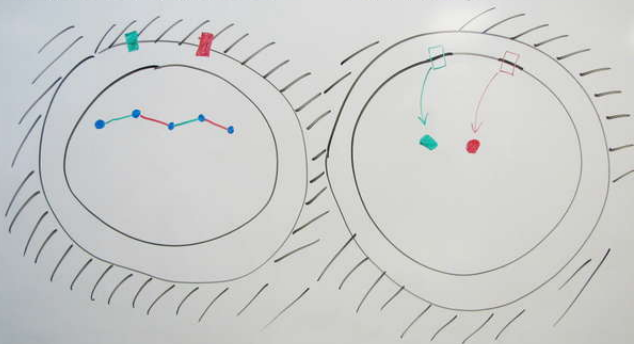
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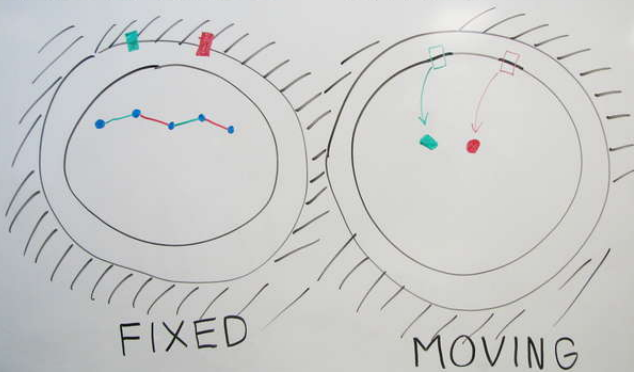
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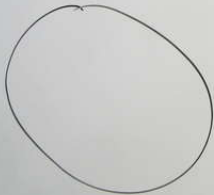
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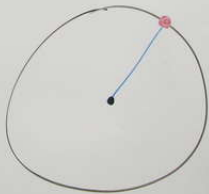
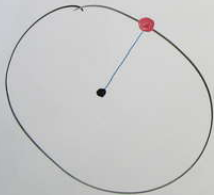
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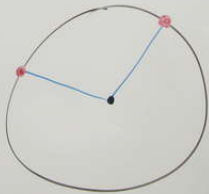
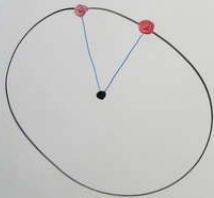
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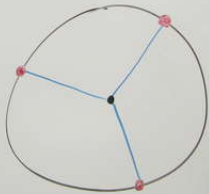
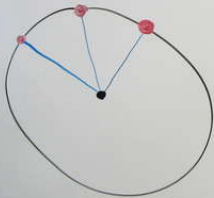
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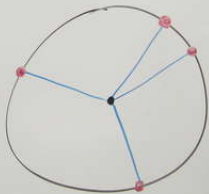
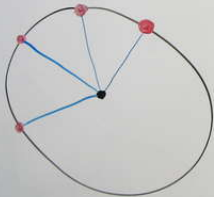
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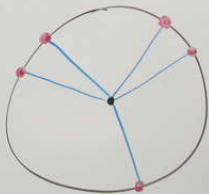
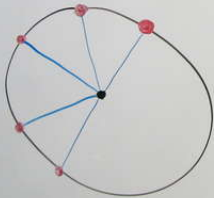
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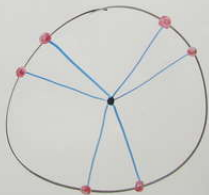
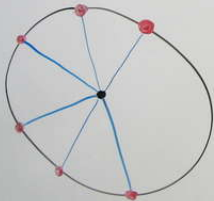
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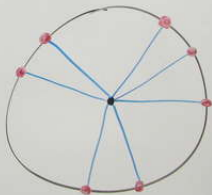
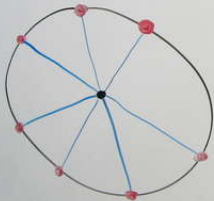
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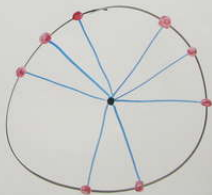
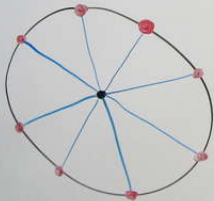
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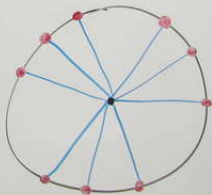
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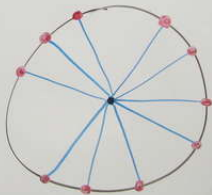
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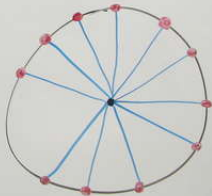
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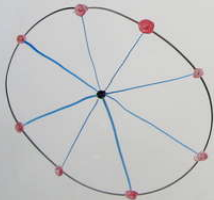
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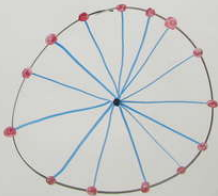
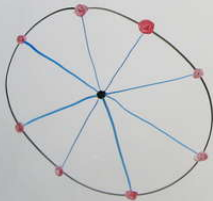
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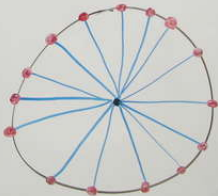
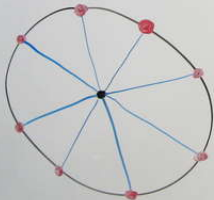
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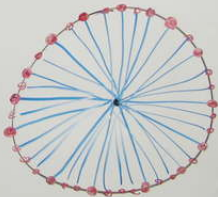
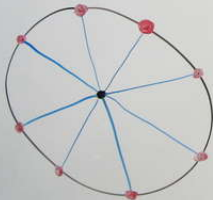
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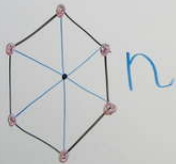
ROTATION ANGLES

RATIONAL / IRRATIONAL
DISCRETE / CONTINUOUS



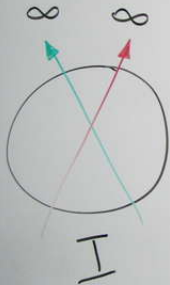
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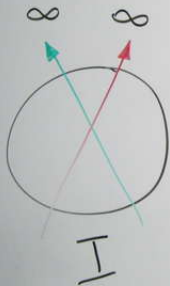


THREE CASES

THREE CASES



THREE CASES



THREE CASES



CASE I (∞, ∞)

CASE I (∞, ∞)

PROOF 1

CASE I (∞, ∞)

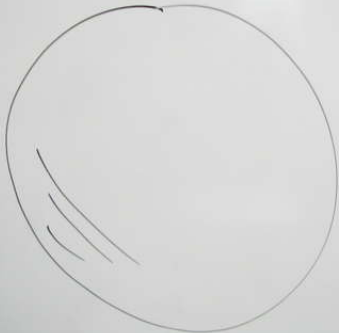
PROOF 1

(moving
reference
frame)

CASE I (∞, ∞)

PROOF 1

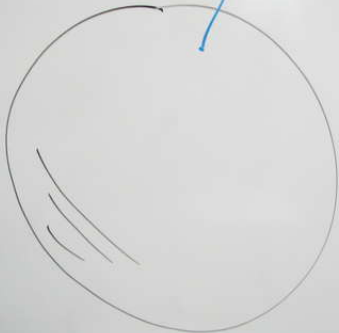
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CASE I (∞, ∞) \odot

PROOF 1

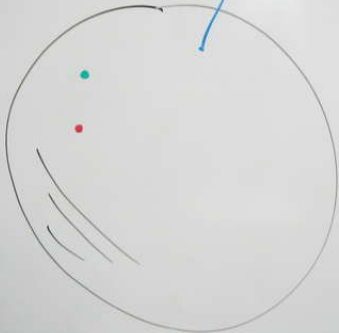
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CASE I (∞, ∞) ∞

PROOF 1

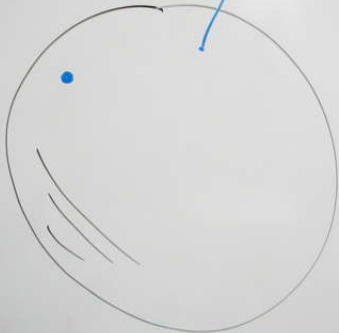
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CASE I (∞, ∞) ω

PROOF 1

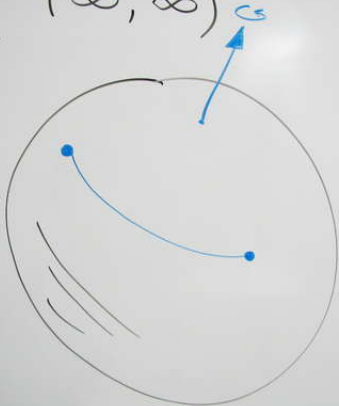
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reference
frame)



CASE I (∞, ∞)

PROOF 1

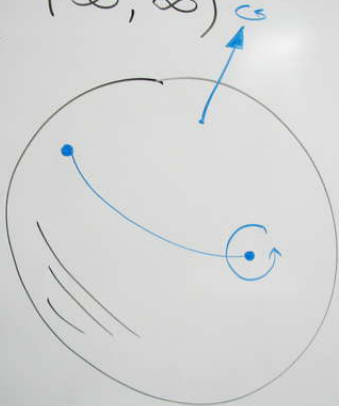
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CASE I (∞, ∞)

PROOF 1

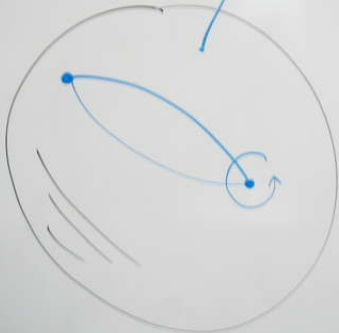
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CASE I (∞, ∞)

PROOF 1

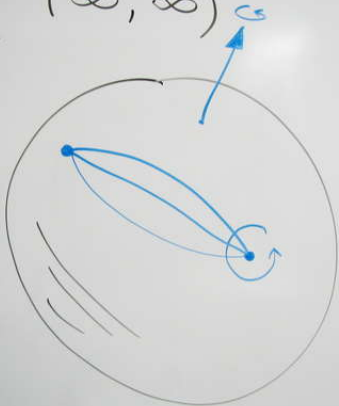
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

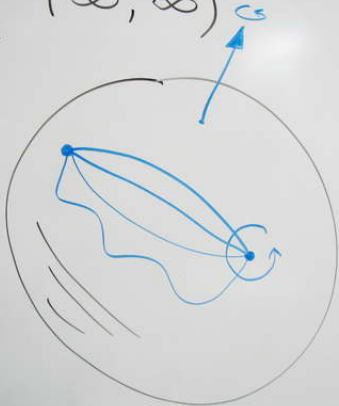
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reference
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CASE I (∞, ∞)

PROOF 1

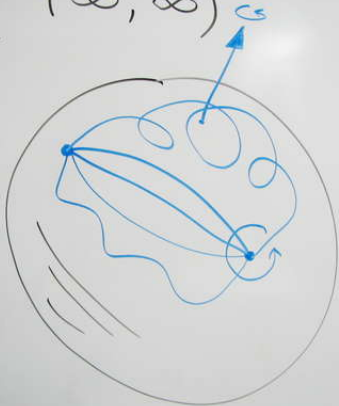
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CASE I (∞, ∞)

PROOF 1

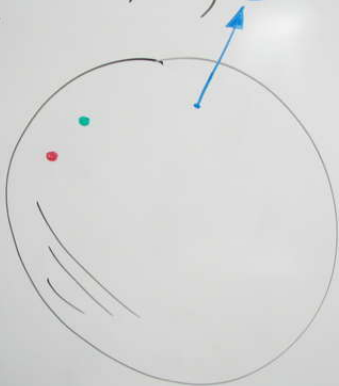
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reference
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CASE I (∞, ∞)

PROOF 1

(moving
reference
frame)

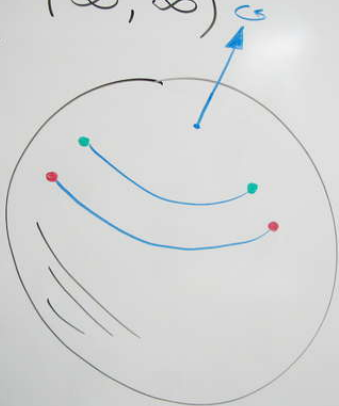


CASE I

(∞, ∞)

PROOF 1

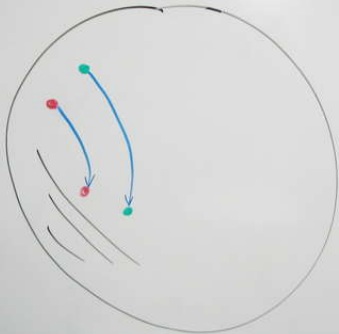
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

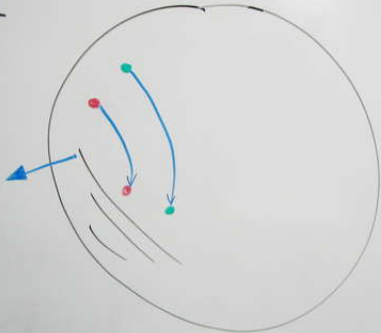
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

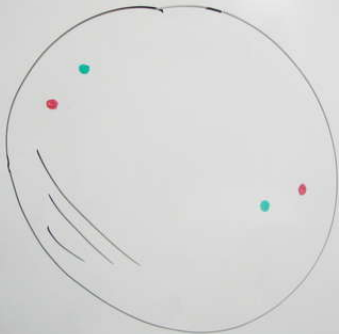
(moving
reference
frame)



CASE I (∞, ∞)

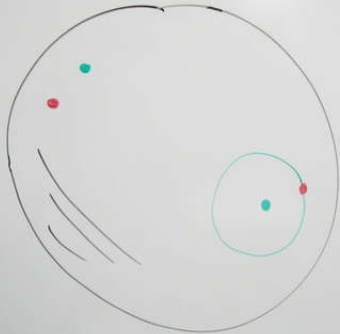
PROOF 1

(moving
reference
frame)



CASE I (∞, ∞)

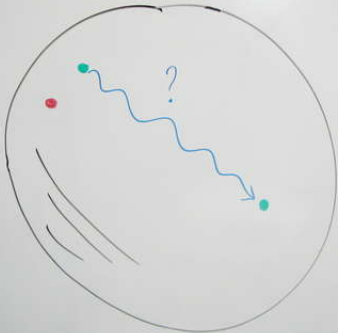
PROOF 1
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

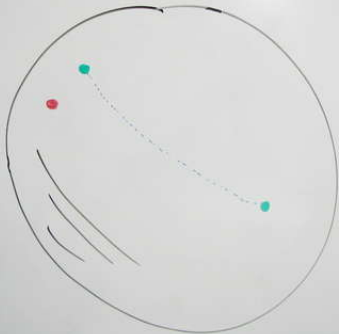
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

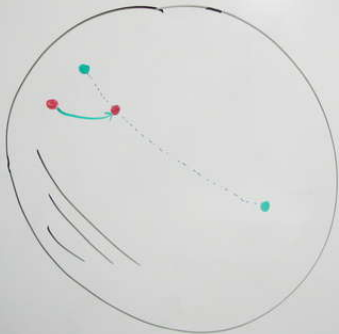
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

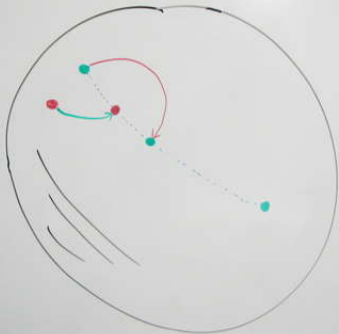
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

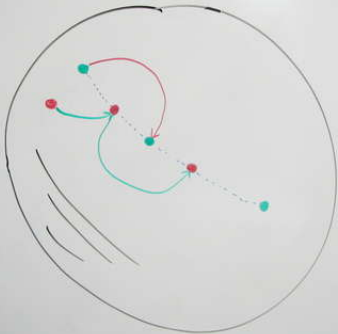
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

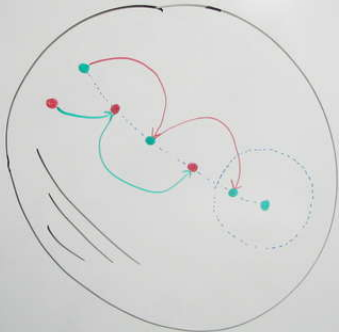
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

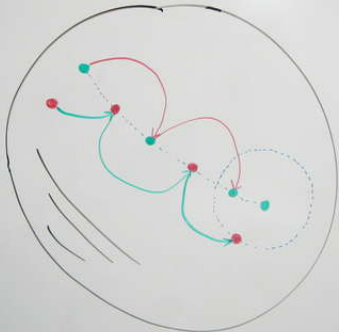
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

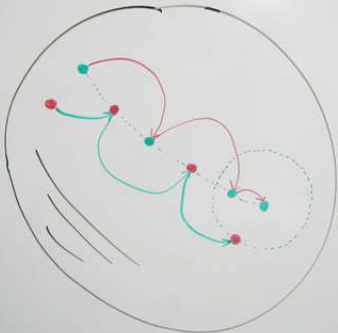
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

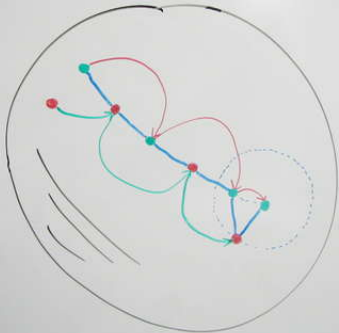
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

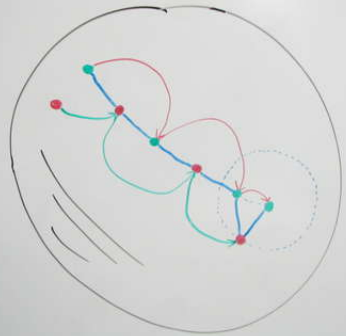
(moving
reference
frame)



CASE I (∞, ∞)

PROOF 1

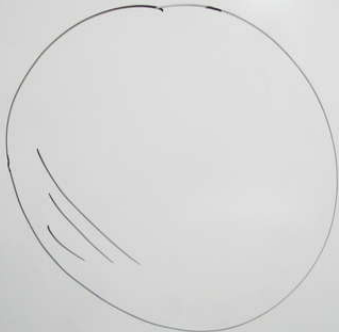
(moving
reference
frame)



CASE I (∞, ∞) $\theta = \frac{\pi}{2}$?

PROOF 1

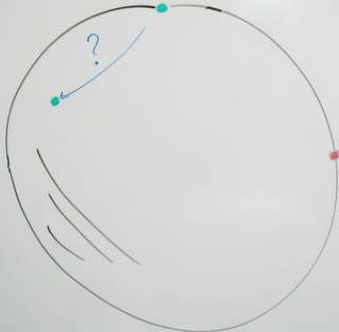
(moving
reference
frame)



CASE I $(-\infty, \infty)$ $\theta = \frac{\pi}{2}$?

PROOF 1

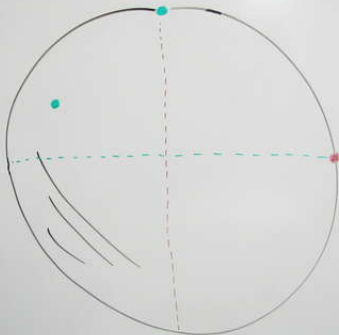
(moving
reference
frame)



CASE I (∞, ∞) $\theta = \frac{\pi}{2}$?

PROOF 1

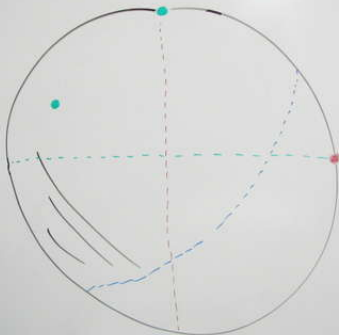
(moving
reference
frame)



CASE I (∞, ∞) $\theta = \frac{\pi}{2}$?

PROOF 1

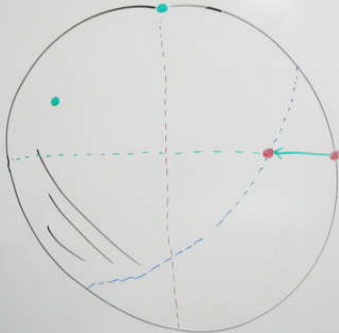
(moving
reference
frame)



CASE I (∞, ∞) $\theta = \frac{\pi}{2}$?

PROOF 1

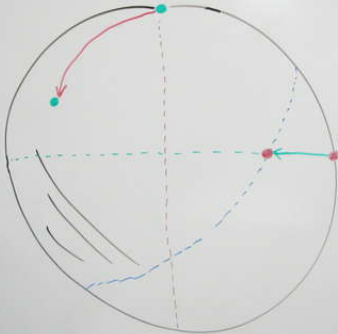
(moving
reference
frame)



CASE I (∞, ∞) $\theta = \frac{\pi}{2}$?

PROOF 1

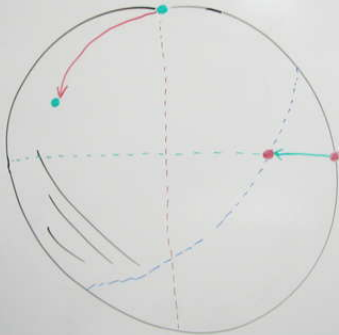
(moving
reference
frame)



CASE I (∞, ∞) $\theta = \frac{\pi}{2}$ OK

PROOF 1

(moving
reference
frame)



CASE I (∞, ∞)

PROOF 2







CASE I (∞, ∞)

PROOF 2

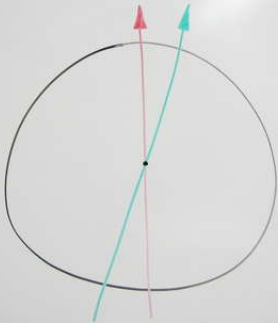
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

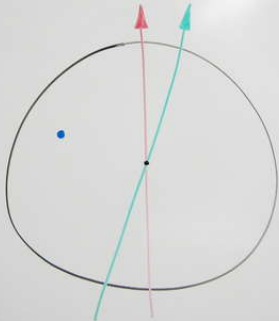
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

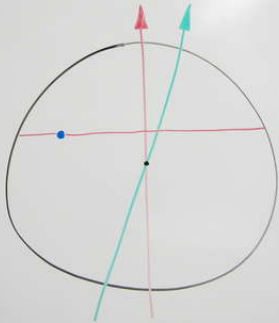
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

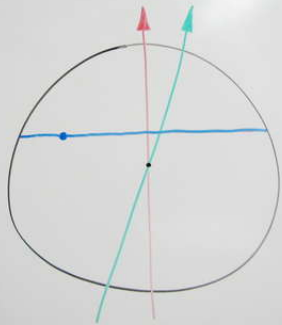
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

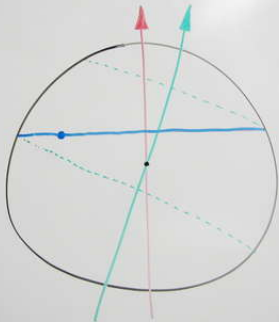
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

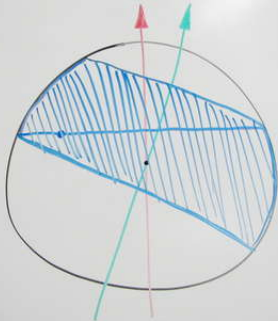
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

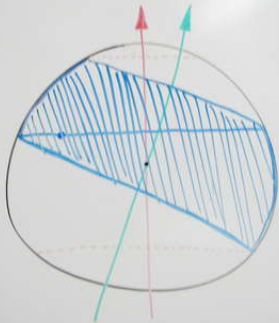
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

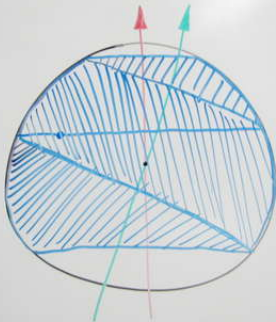
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

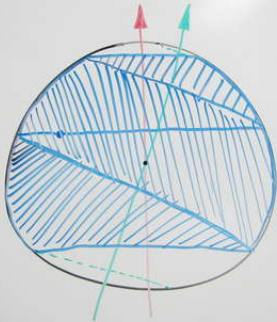
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

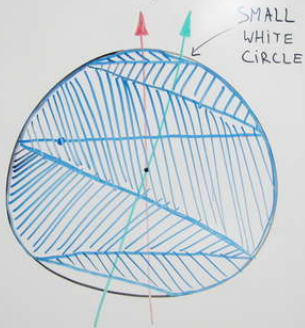
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

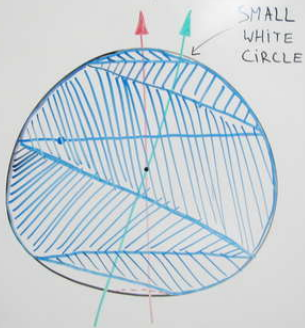
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

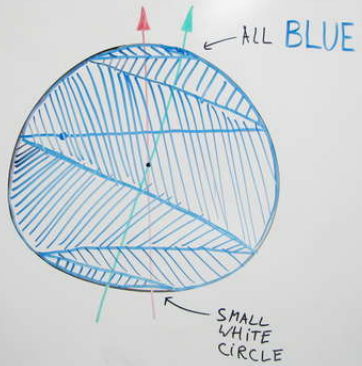
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

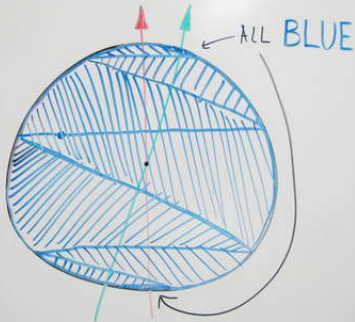
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

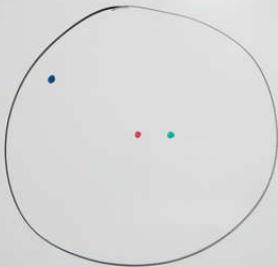
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

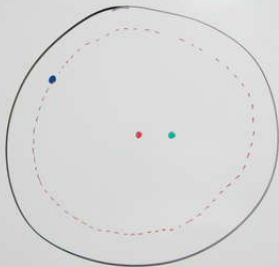
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

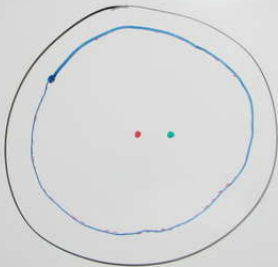
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

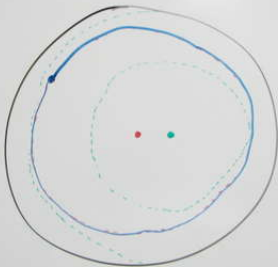
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

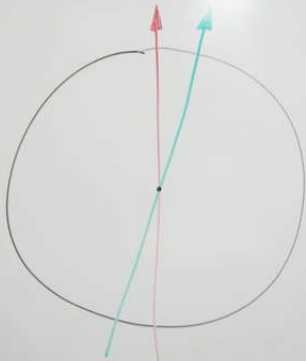
(fixed
reference
frame)

How LONG
IT TAKES?

CASE I (∞, ∞)

PROOF 2

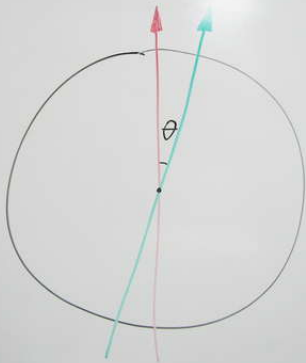
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

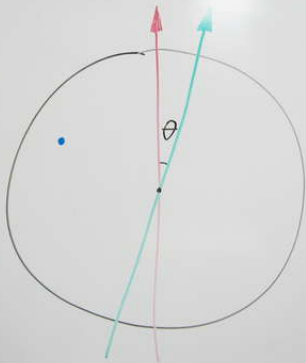
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

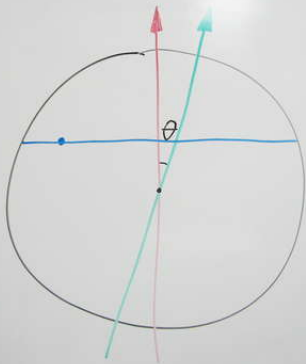
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

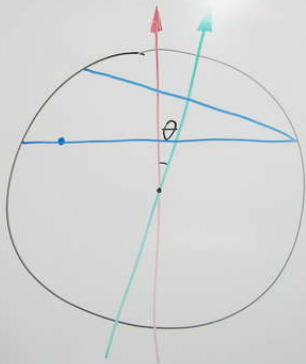
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

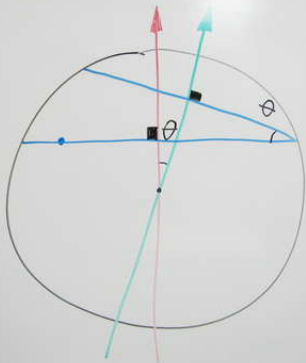
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

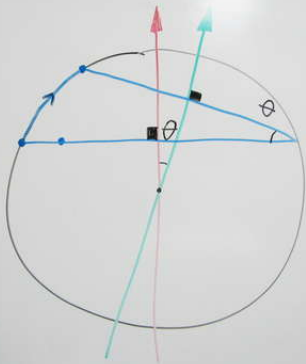
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

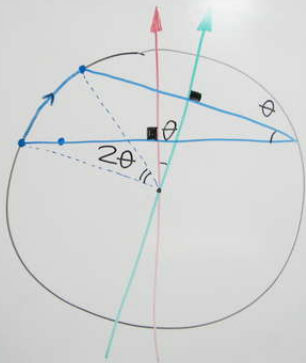
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

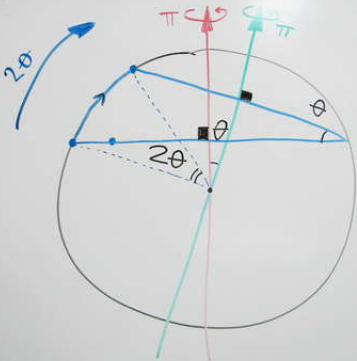
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

(fixed reference frame)

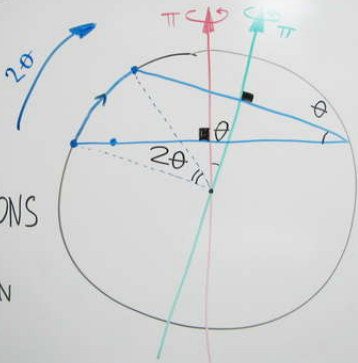


CASE I (∞, ∞)

PROOF 2

(fixed
reference
frame)

2 REFLECTIONS
|||
ROTATION



CASE I (∞, ∞)

PROOF 2

(fixed
reference
frame)

THAT'S NOT
ALL!

CASE I (∞, ∞)

PROOF 2

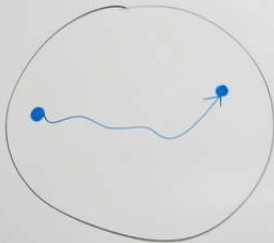
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

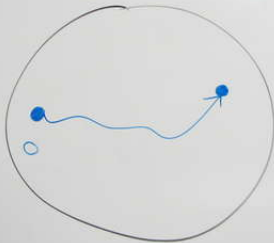
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

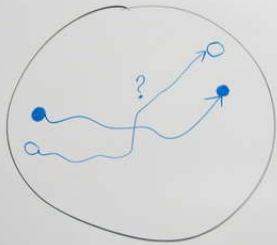
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

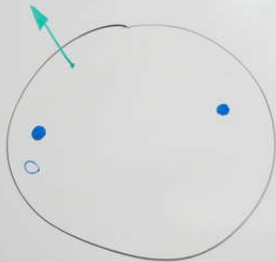
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

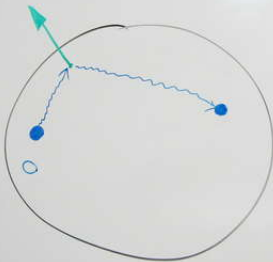
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

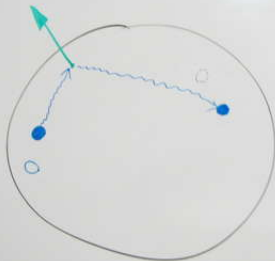
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

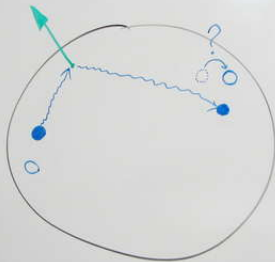
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

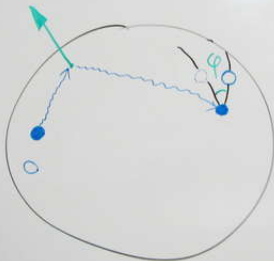
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

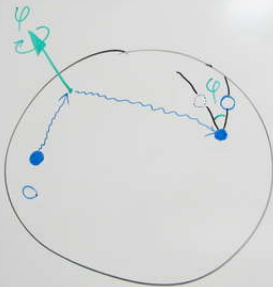
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

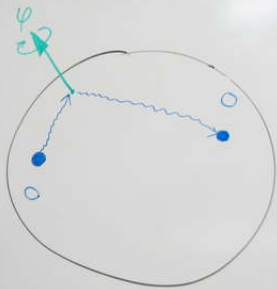
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

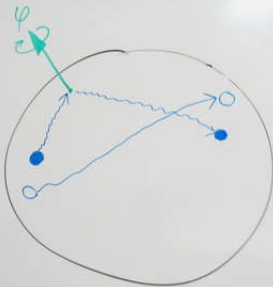
(fixed
reference
frame)



CASE I (∞, ∞)

PROOF 2

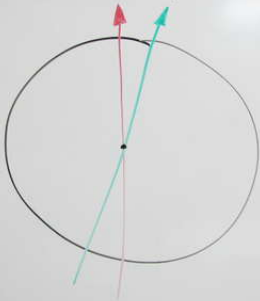
(fixed
reference
frame)



CASE II (∞, n)

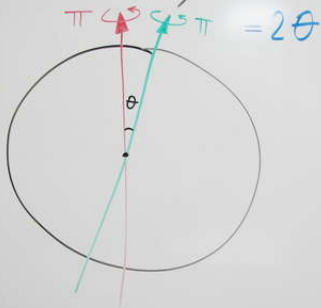
CASE II (∞, n)

If n even:



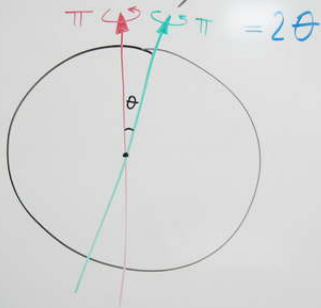
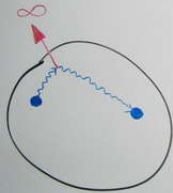
CASE II (∞, n)

If n even:



CASE II (∞, n)

If n even:



CASE II (∞, n)

For any n :

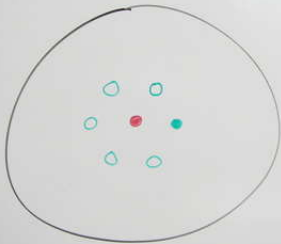
CASE II (∞, n)

For any n :



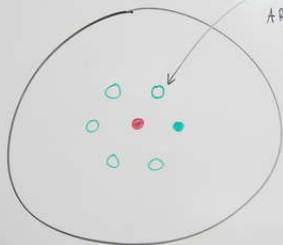
CASE II (∞, n)

For any n :



CASE II (∞, n)

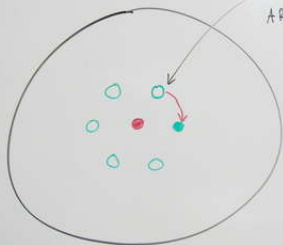
For any n :



CAN WE ROTATE
AROUND THIS
AXIS?

CASE II (∞, n)

For any n :

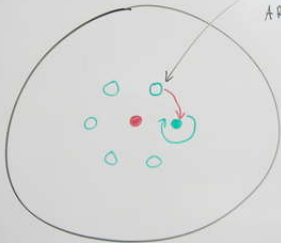


CAN WE ROTATE
AROUND THIS
AXIS?

n

CASE II (∞, n)

For any n :

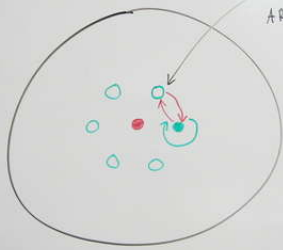


CAN WE ROTATE
AROUND THIS
AXIS?

$n \infty$

CASE II (∞, n)

For any n :

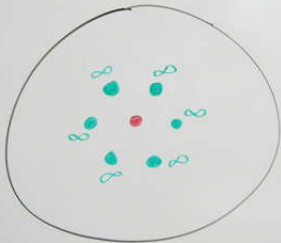


CAN WE ROTATE
AROUND THIS
AXIS?

$$n \infty n^{-1}$$

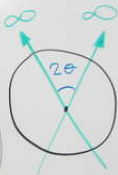
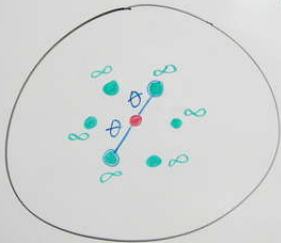
CASE II (∞, n)

For any n :



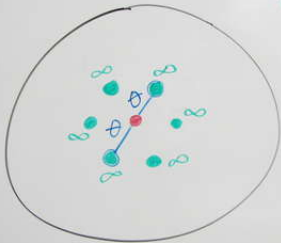
CASE II (∞, n)

For any n :



CASE II (∞, n)

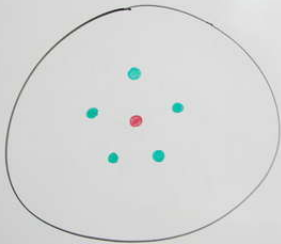
For any n :



USE
CASE I!

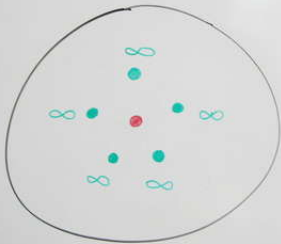
CASE II (∞, n)

For any n :



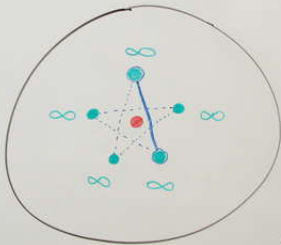
CASE II (∞, n)

For any n :

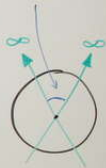


CASE II (∞, n)

For any n :



Almost 2θ



USE
CASE I!

CASE III (n, m)

CASE III (n, m)

WHEN IS THE
GROUP FINITE?

CASE III (n, m)

WHEN IS THE
GROUP FINITE?

- C_k - CYCLIC

CASE III (n, m)

• C_k - CYCLIC



WHEN IS THE
GROUP FINITE?

CASE III (n, m)

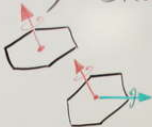
WHEN IS THE GROUP FINITE?

- C_k - CYCLIC
- D_k - DIHEDRAL



CASE III (n, m) WHEN IS THE GROUP FINITE?

- C_k - CYCLIC
- D_k - DIHEDRAL



CASE III (n, m)

WHEN IS THE GROUP FINITE?

- C_k - CYCLIC
- D_k - DIHEDRAL



$$\theta = 0$$



$$\theta = \frac{\pi}{2}$$

$$n=2 \text{ OR } m=2$$

CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL



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CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL



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2 3 3

CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL
- O - OCTAHEDRAL



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2 3 3

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$$\theta = \frac{\pi}{2}$$

n=2 OR m=2



2 3 3



2 4 3



CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL
- O - OCTAHEDRAL



$\theta = 0$



$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$



2 3 3



2 4 3



CASE III (n, m)

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DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL
- O - OCTAHEDRAL



$\theta = 0$



$\theta = \frac{\pi}{2}$
 $n=2$ OR $m=2$



2 3 3



2 4 3



2 3 4

CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL
- O - OCTAHEDRAL
- I - ICOSAHEDRAL



$\theta = 0$



$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$



2 3 3



2 4 3



2 3 4

CASE III (n, m)

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DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL



$$\theta = 0$$



$$\theta = \frac{\pi}{2}$$

$n=2$ OR $m=2$

- O - OCTAHEDRAL



2 3 3



2 4 3



2 3 4

- I - ICOSAHEDRAL



CASE III (n, m)

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- C_k - CYCLIC
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$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$



2 3 3

- O - OCTAHEDRAL



2 4 3



2 3 4

- I - ICOSAHEDRAL



CASE III (n, m)

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- C_k - CYCLIC
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$\theta = 0$



$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

- O - OCTAHEDRAL



2 4 3



2 3 4

- I - ICOSAHEDRAL



2 5 3



2 3 5

CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL



$\theta = 0$



$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

- 12 • T - TETRAHEDRAL



2 3 3

- 24 • O - OCTAHEDRAL



2 4 3



2 3 4

- 60 • I - ICOSAHEDRAL



2 5 3



2 3 5

CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL



$$\theta = 0$$



$$\theta = \frac{\pi}{2}$$

$$n=2 \text{ OR } m=2$$

- FINITE NUMBER
- 12 • T - TETRAHEDRAL
 - 24 • O - OCTAHEDRAL



2 3 3



2 4 3



2 3 4

- 60 • I - ICOSAHEDRAL



2 5 3



2 3 5

CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL



$\theta = 0$



$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

FINITE NUMBER

- 12 • O - OCTAHEDRAL
- 24 • I - ICOSAHEDRAL



2 3 3



2 4 3



2 3 4



2 5 3



2 3 5

CASE III (n, m) WHEN IS THE GROUP

CASE III (n, m)

WHEN IS THE
GROUP INFINITE?

CASE III (n, m) WHEN IS THE GROUP INFINITE?

IN ALL OTHER CASES.

CASE III (n, m) WHEN IS THE GROUP INFINITE?

IN ALL OTHER CASES.

HOW DO WE GET AN
IRRATIONAL ANGLE OUT OF
TWO RATIONAL ANGLES?

CASE III (n, m) WHEN IS THE GROUP INFINITE?

IN ALL OTHER CASES.

HOW DO WE GET AN
IRRATIONAL ANGLE OUT OF
TWO RATIONAL ANGLES?

THEN WE CAN USE CASE II.

CASE III (n, m)

WHEN IS THE
GROUP INFINITE

IN ALL OTHER CASES.

HOW DO WE GET AN
IRRATIONAL ANGLE OUT OF
TWO RATIONAL ANGLES?

THEN WE CAN USE CASE II.

$$\alpha, 0 - \{ (0, 0), (1, \sqrt{3} + \alpha) \}, (2, \sqrt{3} + 2\alpha)$$

Do not erase!

$$y = x^2 + \alpha x + \beta$$

$$y - x^2 = \alpha x + \beta$$

CASE III (n, m) WHEN IS THE
GROUP INFINITE

IN ALL OTHER CASES.

HOW DO WE GET AN
IRRATIONAL ANGLE OUT OF
TWO RATIONAL ANGLES?

THEN...

Do not erase!

$$y = x^2 + \alpha x + \beta$$
$$y - x^2 = \alpha x + \beta$$

$$\sum_{x \in \mathbb{Z}_p} \langle x, \alpha x + \beta \rangle$$

CASE III (n, m) WHEN IS THE GROUP INFINITE?

IN ALL OTHER CASES.

HOW DO WE GET AN
IRRATIONAL ANGLE OUT OF
TWO RATIONAL ANGLES?

THEN WE CAN USE CASE II.

$$\sum_{\substack{x \in \mathbb{Z} \\ y = x^2 + \alpha x + \beta}} \omega^{x^2 + \alpha x + \beta} |x, x^2 + \alpha x + \beta\rangle$$

$$\rightarrow \sum_{x \in \mathbb{Z}} |x, \alpha x + \beta\rangle \quad P_{\alpha, \beta} = \{ (x, \alpha x + \beta) \}$$

$$P_{\alpha, 0} = \{ (0, 0), (1, \alpha), (2, 2\alpha) \}$$

$$P_{\alpha, 0} = \{ (x, \alpha x) \}$$

Do not erase!

$$\sum_{x \in \mathbb{Z}} |x, \alpha x + \beta\rangle$$

$$|g\rangle = \frac{1}{\sqrt{N}} \sum_x |g x\rangle$$



CASE III (n, m) WHEN IS THE GROUP INFINITE?

IN ALL OTHER CASES.

HOW DO WE GET AN IRRATIONAL ANGLE OUT OF TWO RATIONAL ANGLES?

THEN WE CAN USE CASE II.

$$\sum_y \sum_x \omega^{y^2 + \alpha y} \omega^{x^2 + \alpha x} |y\rangle$$

$$\sum_{\substack{a \in \mathbb{Z} \\ \gcd(a, n) = 1}} w^{a(n+1)} \cdot \underbrace{f_2(a)}_{\text{...}}$$

$$P_{n,0} \{ (0,0), (1, (n+1)), (2, (n+2)) \}$$

Do not erase!

CASE III (n, m) WHEN IS THE GROUP INFINITE?

IN ALL OTHER CASES

HOW DO WE GET AN
RATIONAL POINT OUT OF
TWO RATIONAL POINTS?

THEN WE CAN USE CASE II

$$P_{n,0} \{ (x, x^2) \}$$

$$\sum_{a \in \mathbb{Z}} f_2(a)$$



$$19 \rightarrow \frac{1}{2} \sum_{a \in \mathbb{Z}} f_2(a)$$

$$\frac{1}{2} \sum_{a \in \mathbb{Z}} f_2(a) = \frac{1}{2} \sum_{a \in \mathbb{Z}} w^{a(n+1)}$$



$$\sum_{i=1}^n \frac{1}{i^2} = \frac{\pi^2}{6}$$

$P_{200}(100, (1+\sqrt{2})), (2+\sqrt{2})$

Do not erase!

CASE III (n, m) WHEN IS THE GROUP INFINITE?

IN ALL OTHER CASES

HOW DO WE GET IN
RATIONAL ANSWER OUT OF
IRRATIONAL ANSWER?

THEN WE CAN USE CASE II

$P_{200}(100, (1+\sqrt{2})), (2+\sqrt{2})$
 $\sum_{i=1}^n \frac{1}{i^2}$
 $\sum_{i=1}^n \frac{1}{i^2}$
 $\sum_{i=1}^n \frac{1}{i^2}$
 $\sum_{i=1}^n \frac{1}{i^2}$



(x_1, y_1) (x_2, y_2)
 $(1, 2)$ $(3, 4)$
 $(1, 2)$ $(3, 4)$
 not erase!
 (n, m) WHEN IS THE GROUP FINITE?
 OTHER CASES
 WHEN WE CAN USE CASE II



CLEAN
 YOUR
 DISHES
 RIGHT NOW

Papers and documents on a yellow table in the foreground.





CLEAN
YEAR
DIVERS
DEAD NEW



CLEAN
YOUR
DISHES
RIGHT NOW



CLEAN
YOUR
DISHES
RIGHT NOW



2
3
NOW













600W

5:10

Lipton
Caramel Twist
Two billion & counting






THE END

3:08

Instant Noodles

THE END

51

A close-up photograph of a hand holding a white rectangular piece of paper inside a microwave oven. The paper is held in the center of the microwave's interior, which is lined with a light-colored, textured material. The words "THE END" are written on the paper in a simple, black, sans-serif font. The microwave door is open, and the interior is brightly lit. The hand is visible on the left side of the frame, holding the paper from the bottom-left corner. The microwave's interior has a circular turntable visible in the background.

THE END

THE END

THE END

THE END



