Universal Pairs of Rotations

Maris Ozols, Laura Mancinska,
Andrew Childs, Debbie Leung

Abstract

Assume you are given a sphere in three dimensions and a pair of rotations about two fixed axis by two fixed angles. Can you compose these rotations to approximate any given rotation to any desired accuracy?

We will characterize the cases when it is possible and show how one should compose the two rotations to approximate an arbitrary rotation.

UNIVERSAL

Universal Pairs

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of

Universal Pairs of Rotations

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~2008~

ARE 3 AXES ENOUGH?

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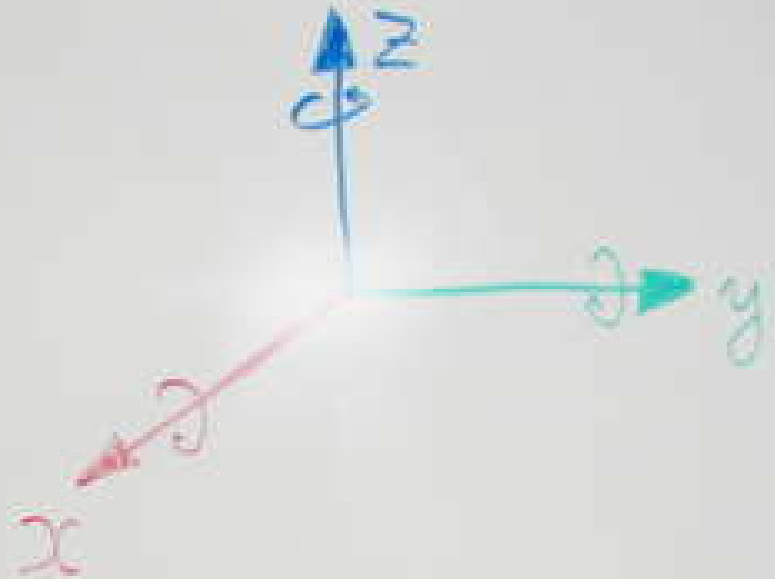

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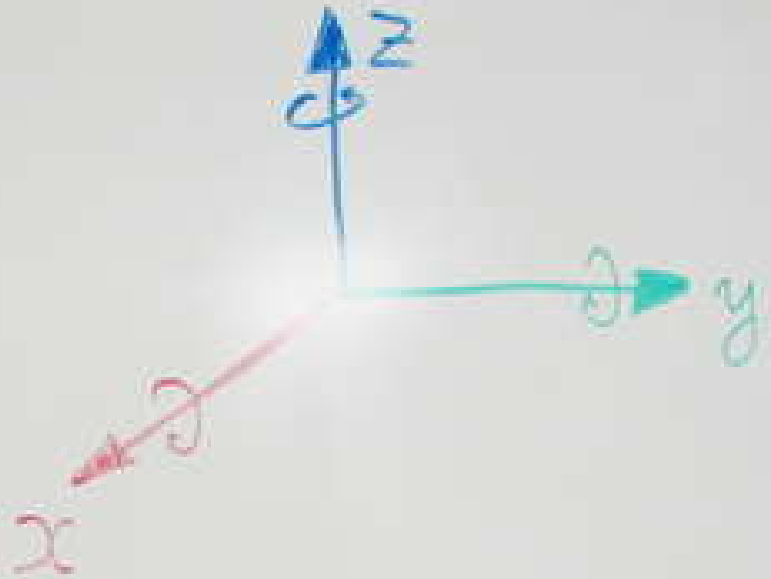

ARE 2 AXES ENOUGH?

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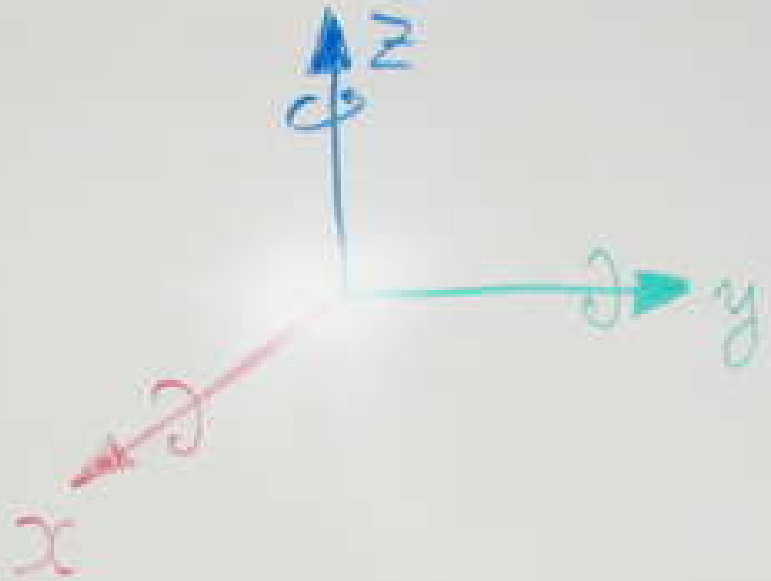

ARE 2 AXES ENOUGH?

PROBLEM:

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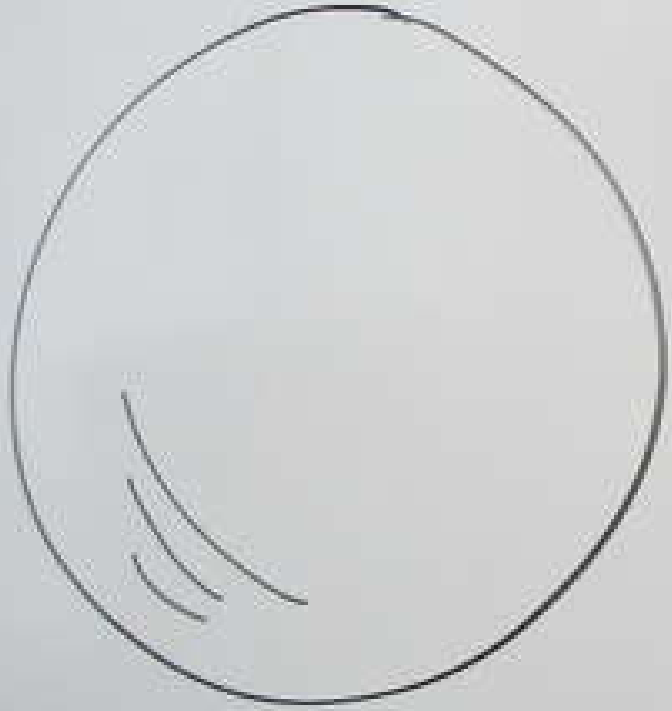

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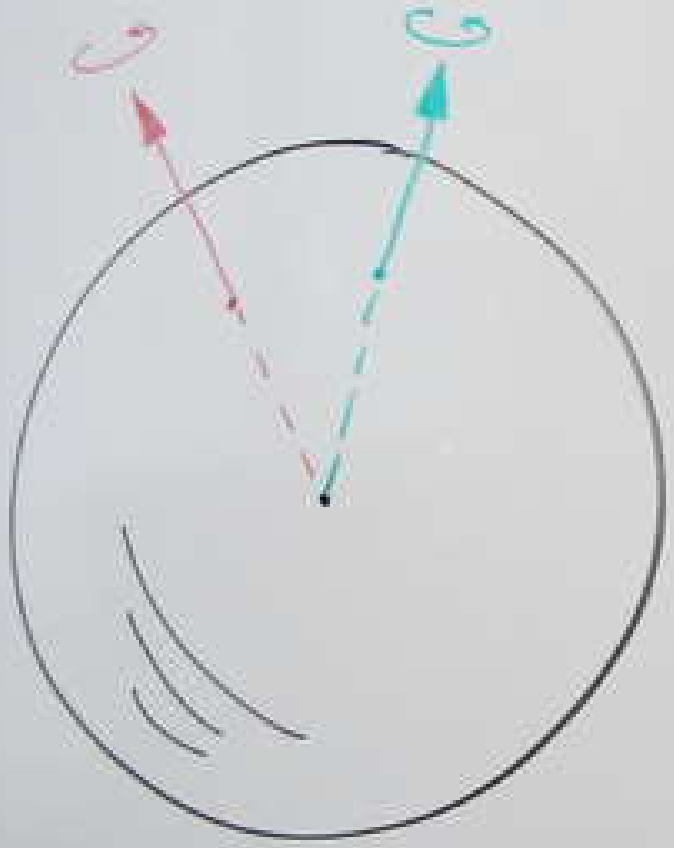

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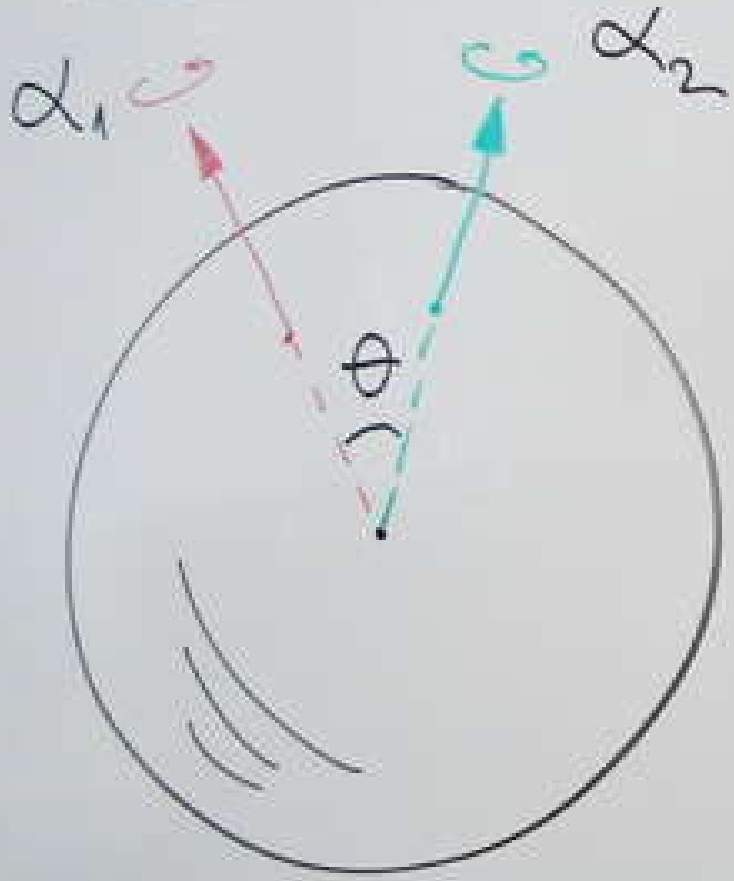

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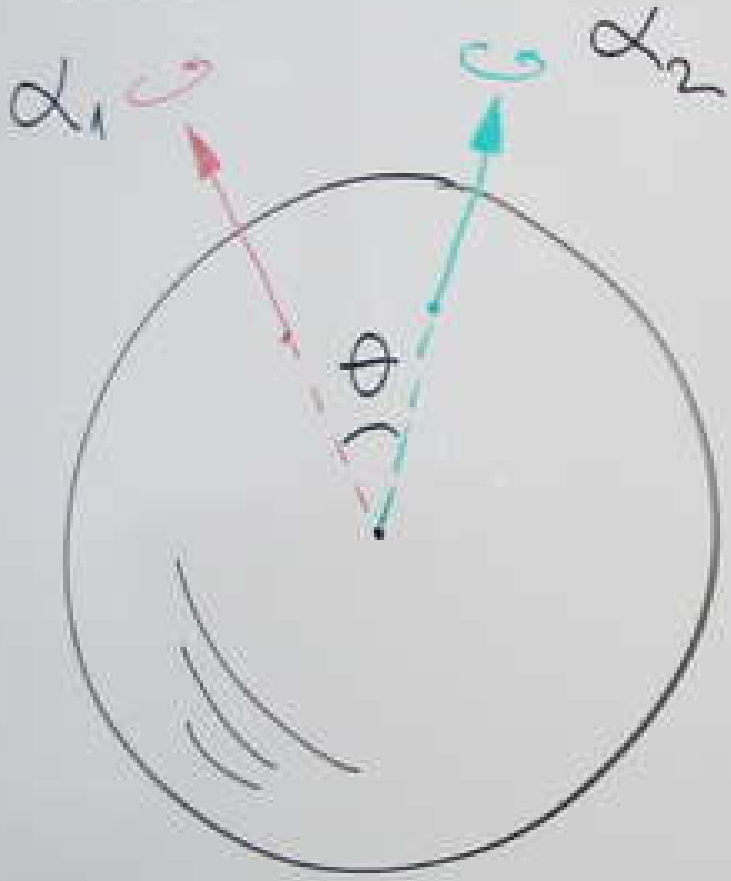

IS THE PAIR OF ROTATIONS $(\theta, \alpha_1, \alpha_2)$ UNIVERSAL

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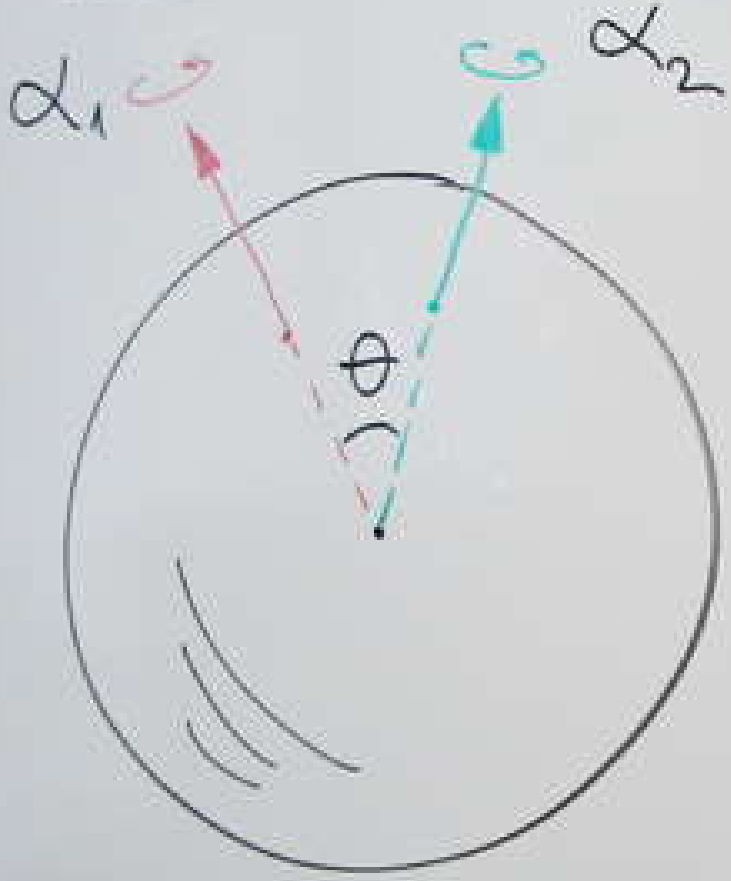

IS THE PAIR OF ROTATIONS $(\theta, \alpha_1, \alpha_2)$ UNIVERSAL (CAN APPROXIMATE ANY ROTATION)

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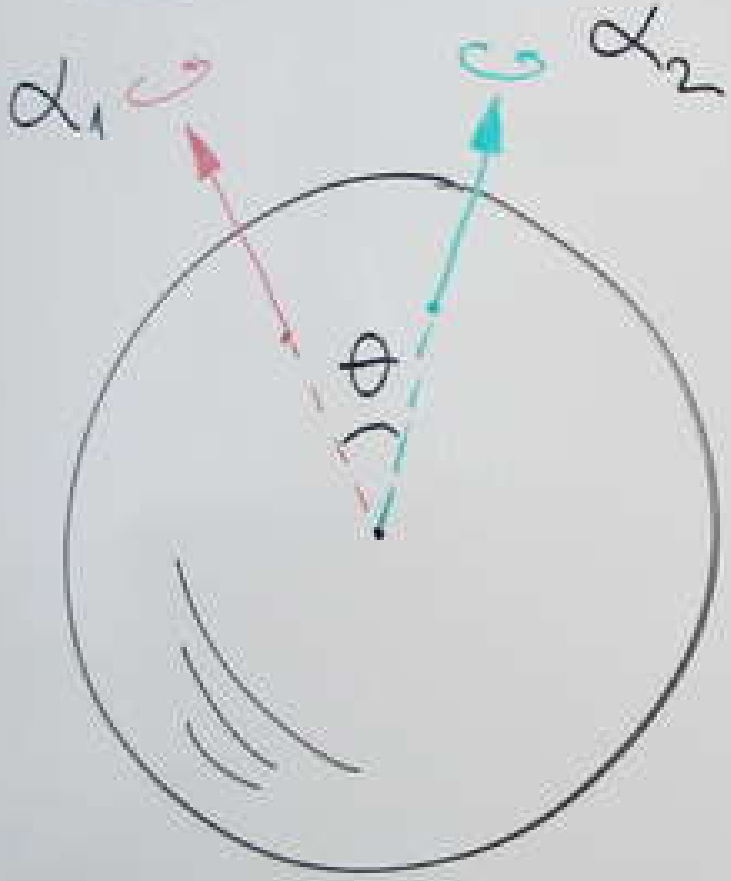

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$(\theta, \alpha_1, \alpha_2)$

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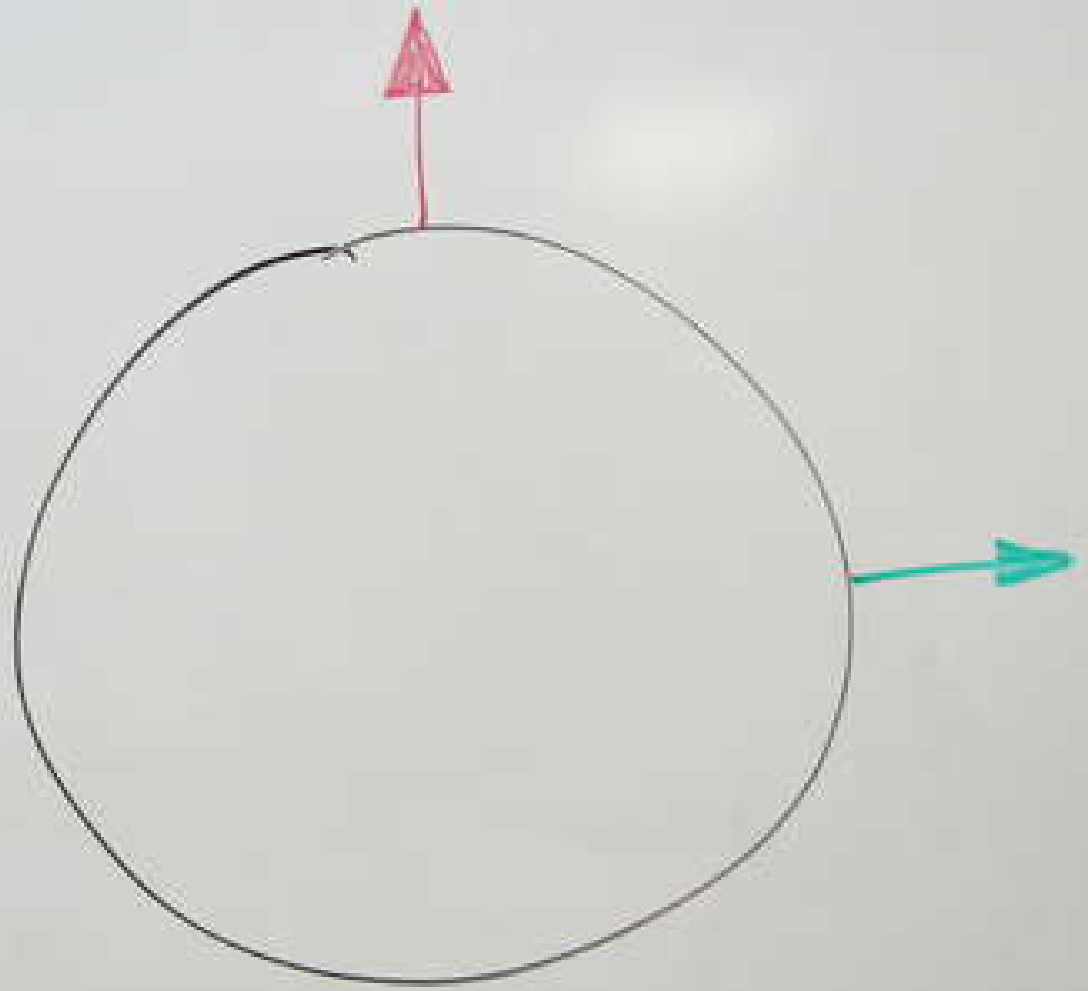
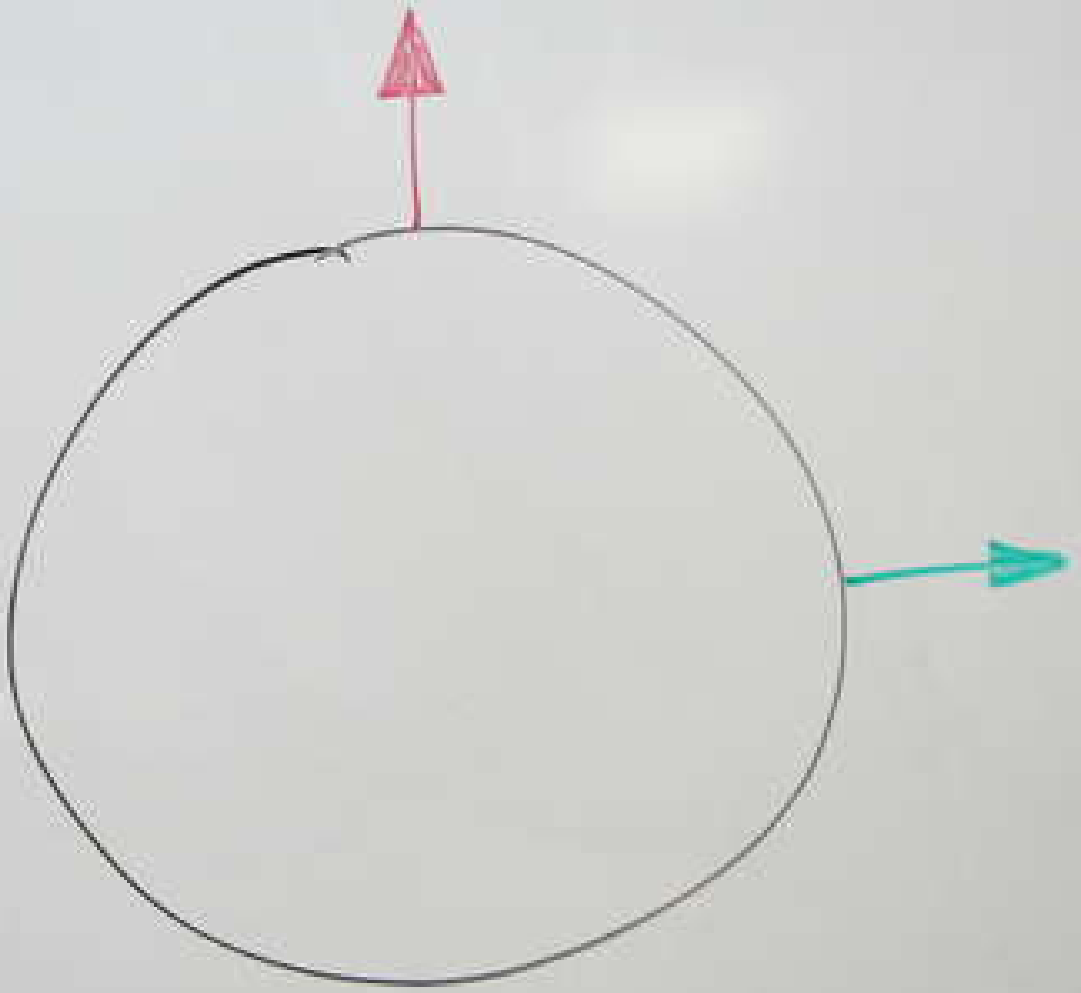


LET'S TRY ...

$(\theta, \alpha_1, \alpha_2)$

$= \frac{\pi}{2}$



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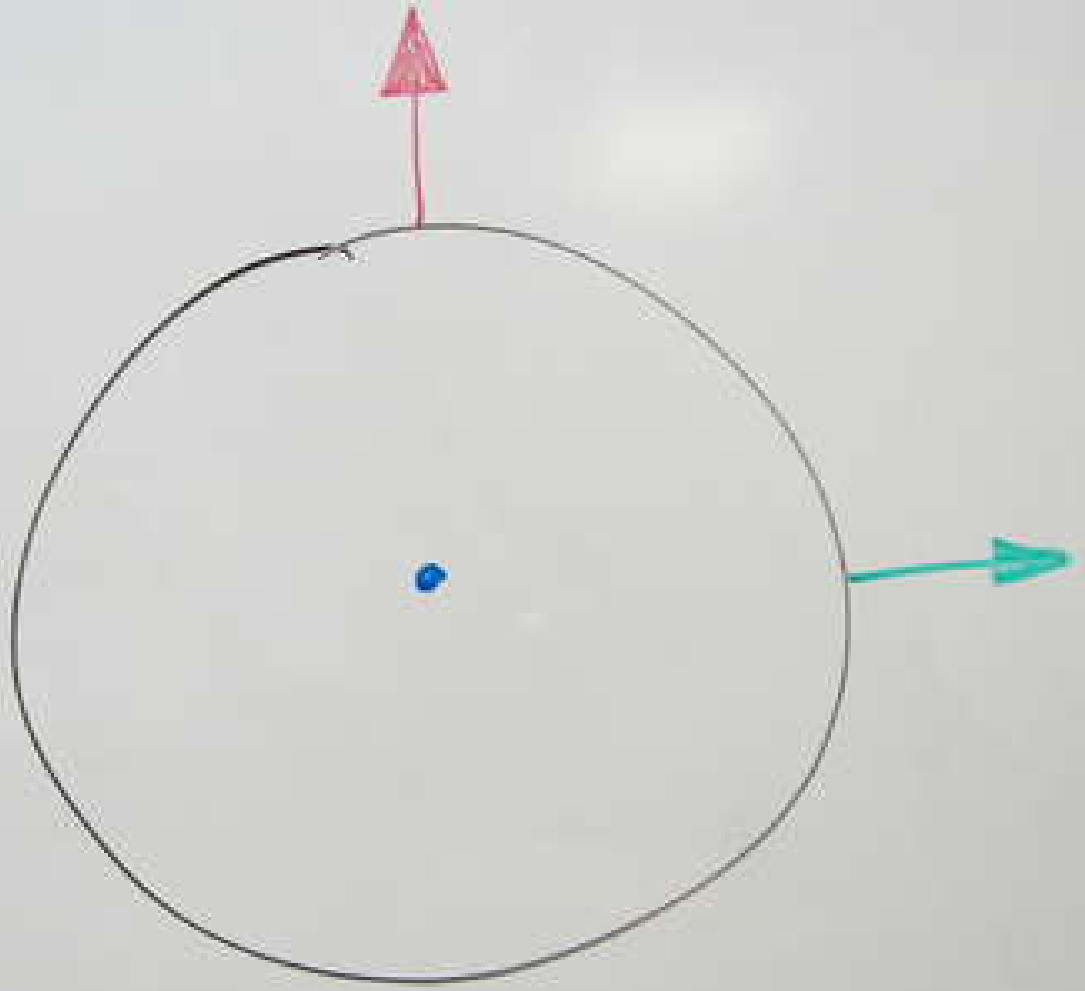

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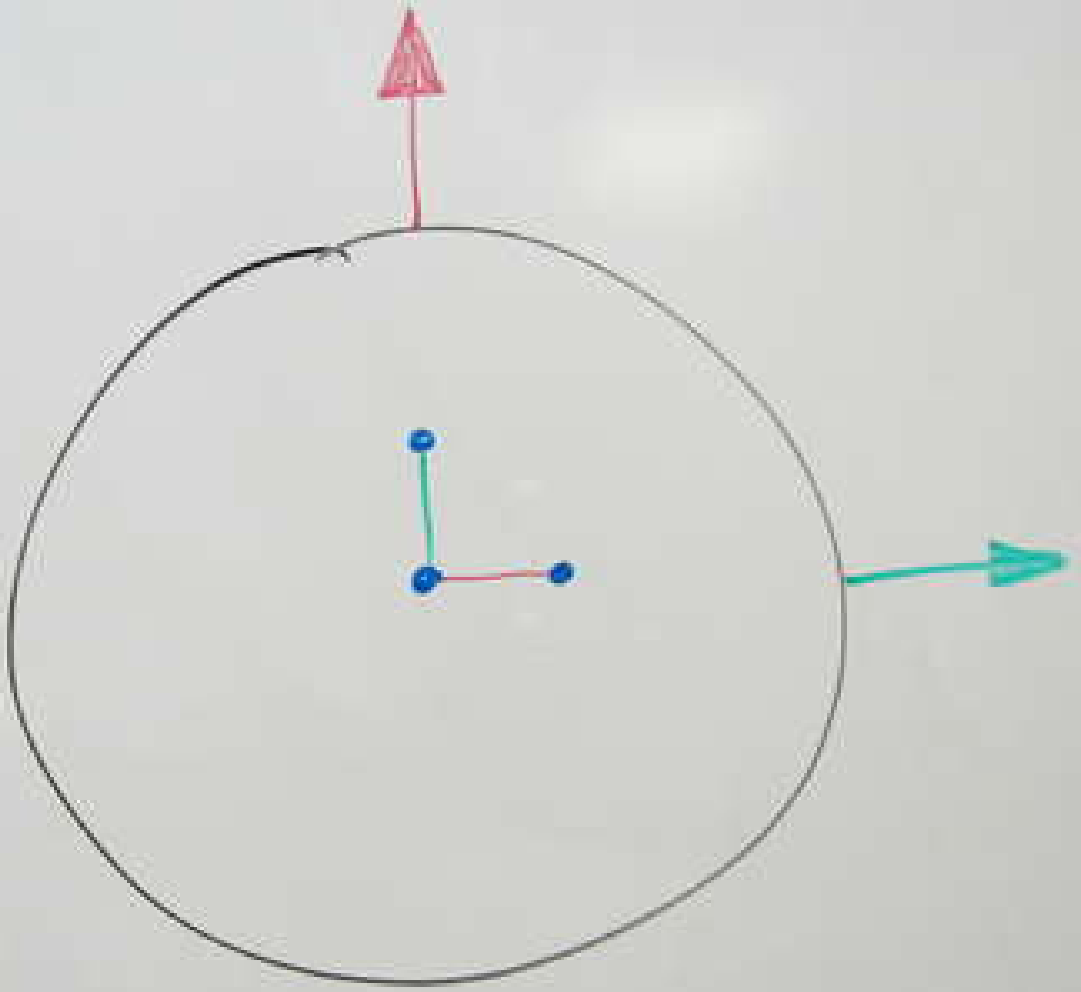

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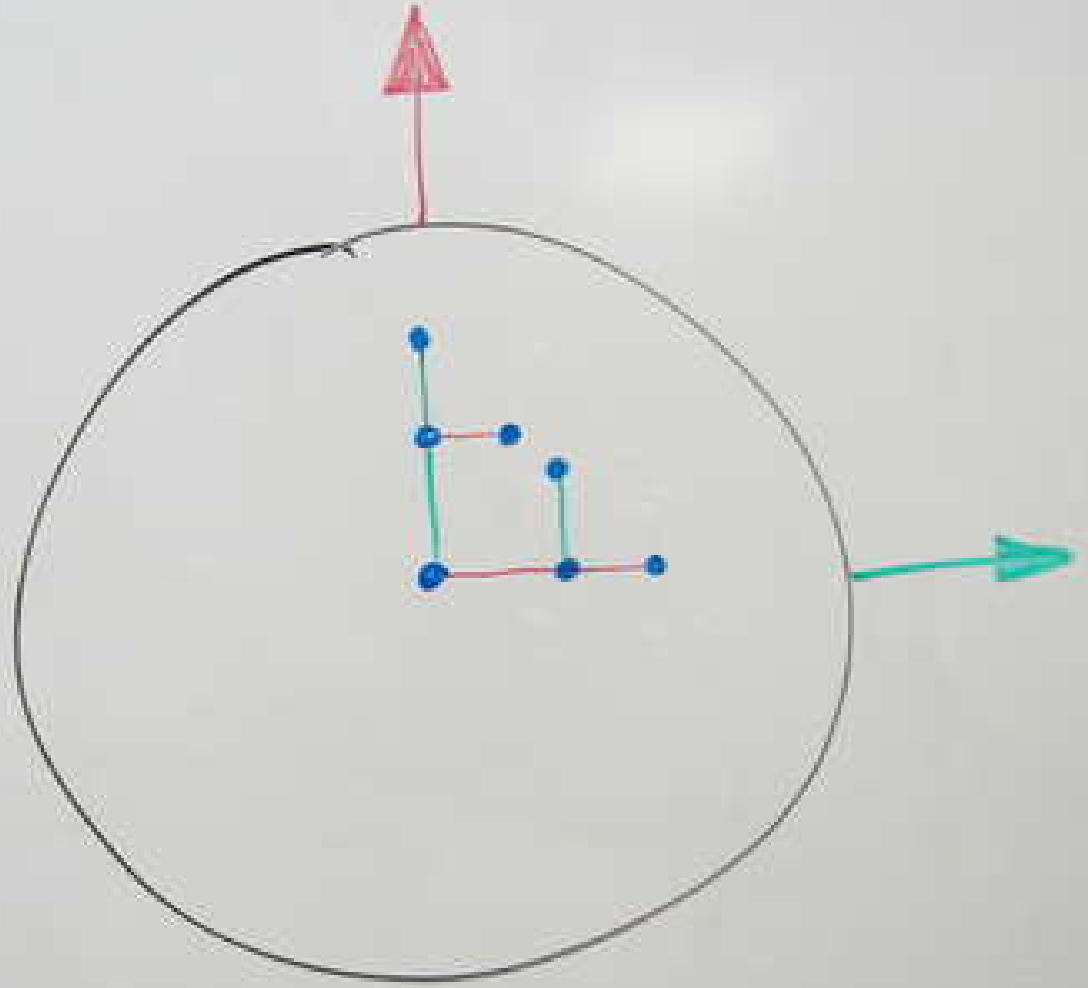

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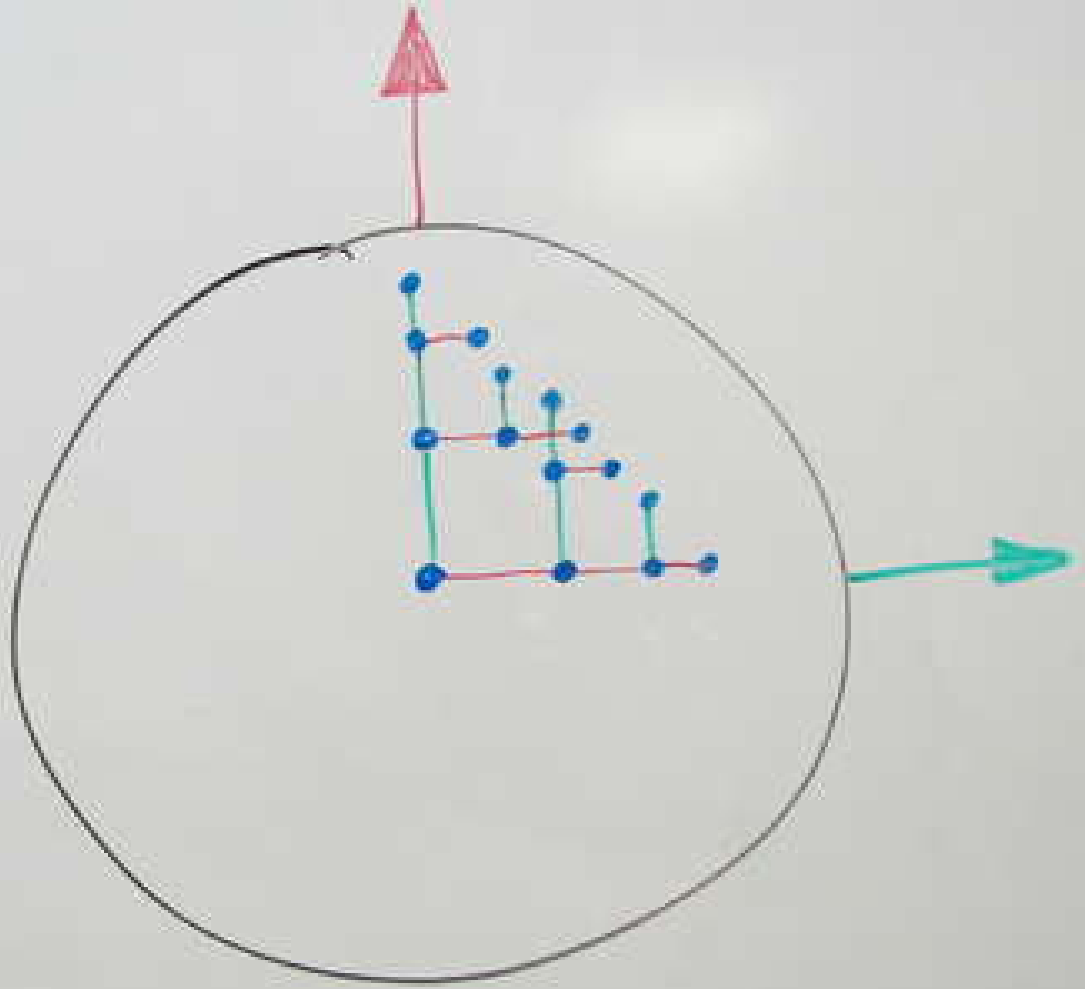

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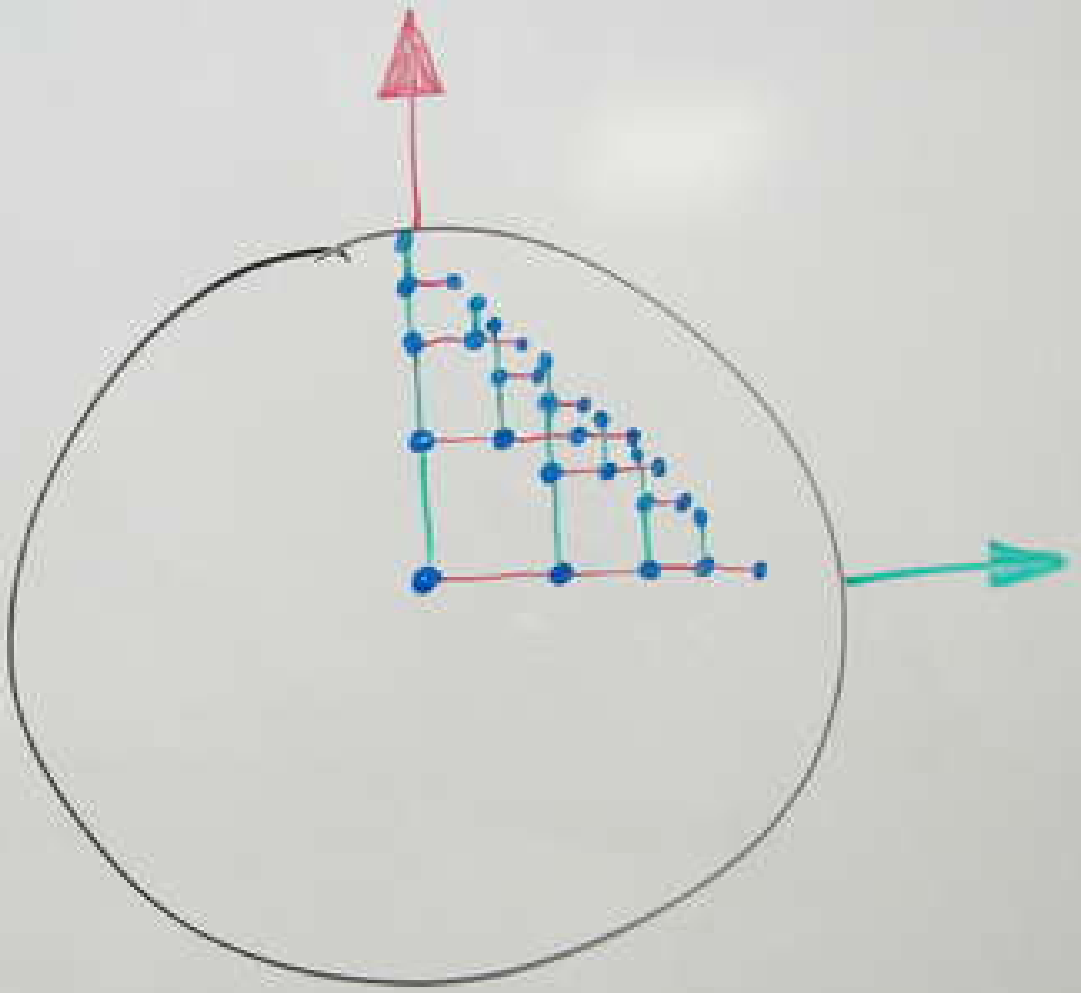
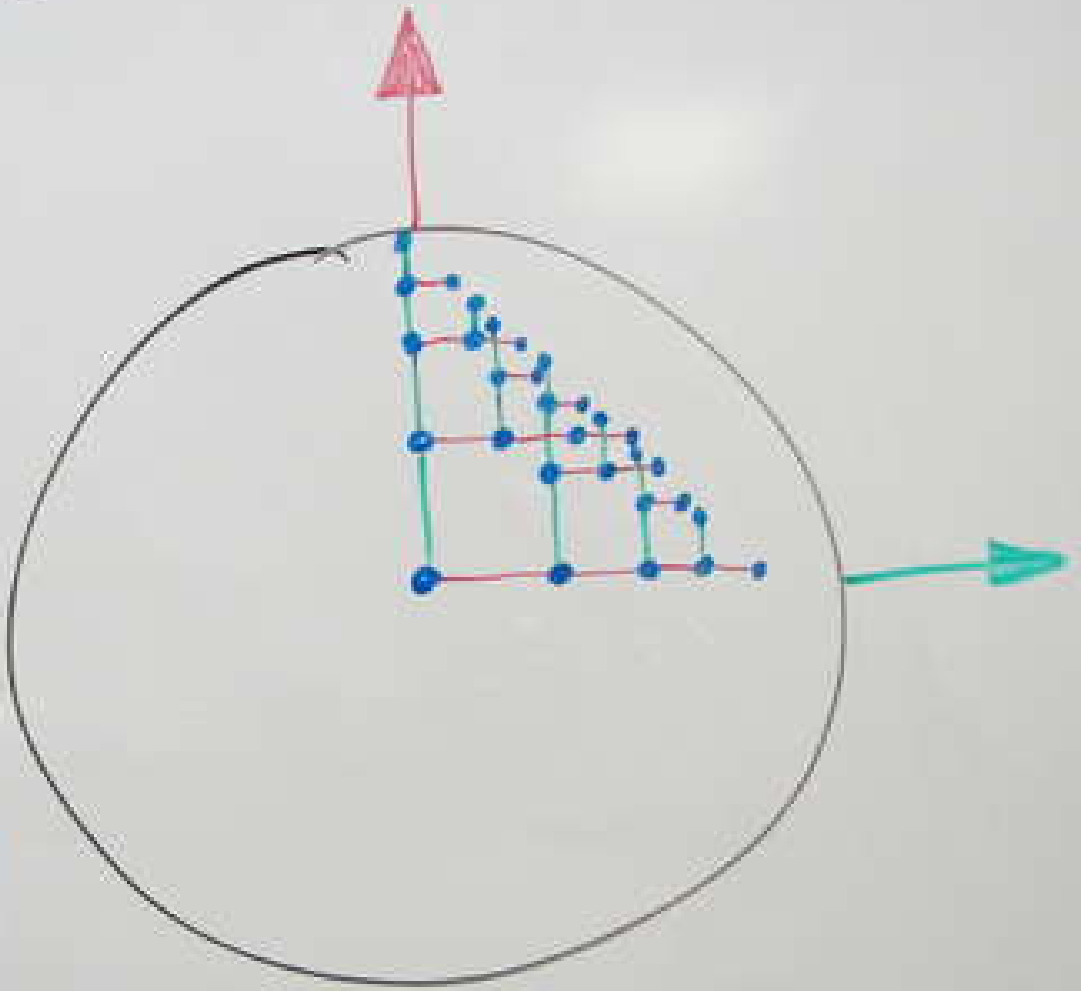


LET'S TRY ...

$(\theta, \alpha_1, \alpha_2)$

$\theta = \frac{\pi}{2}$



WHAT'S GOING ON?

REFERENCE FRAMES

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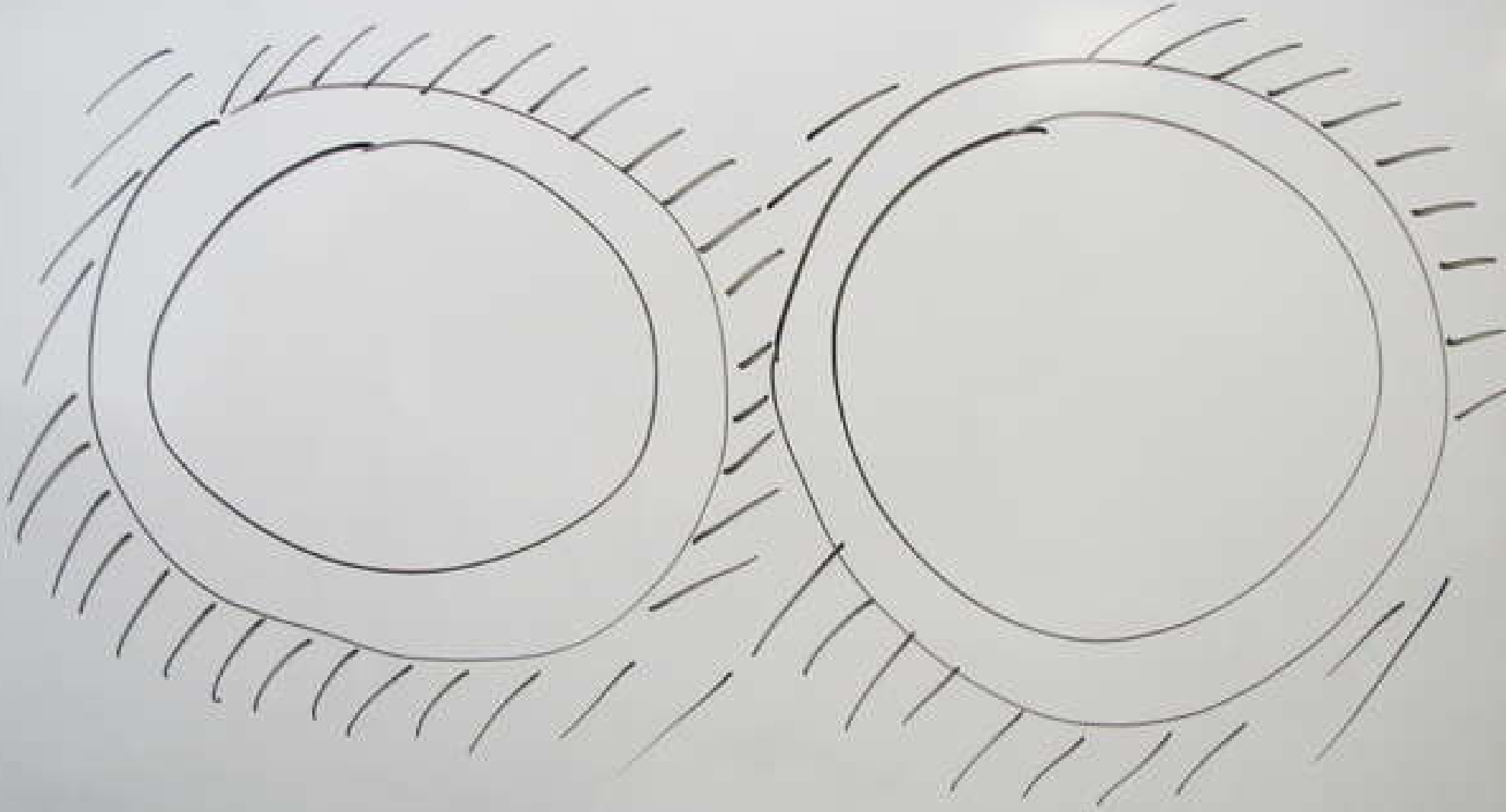

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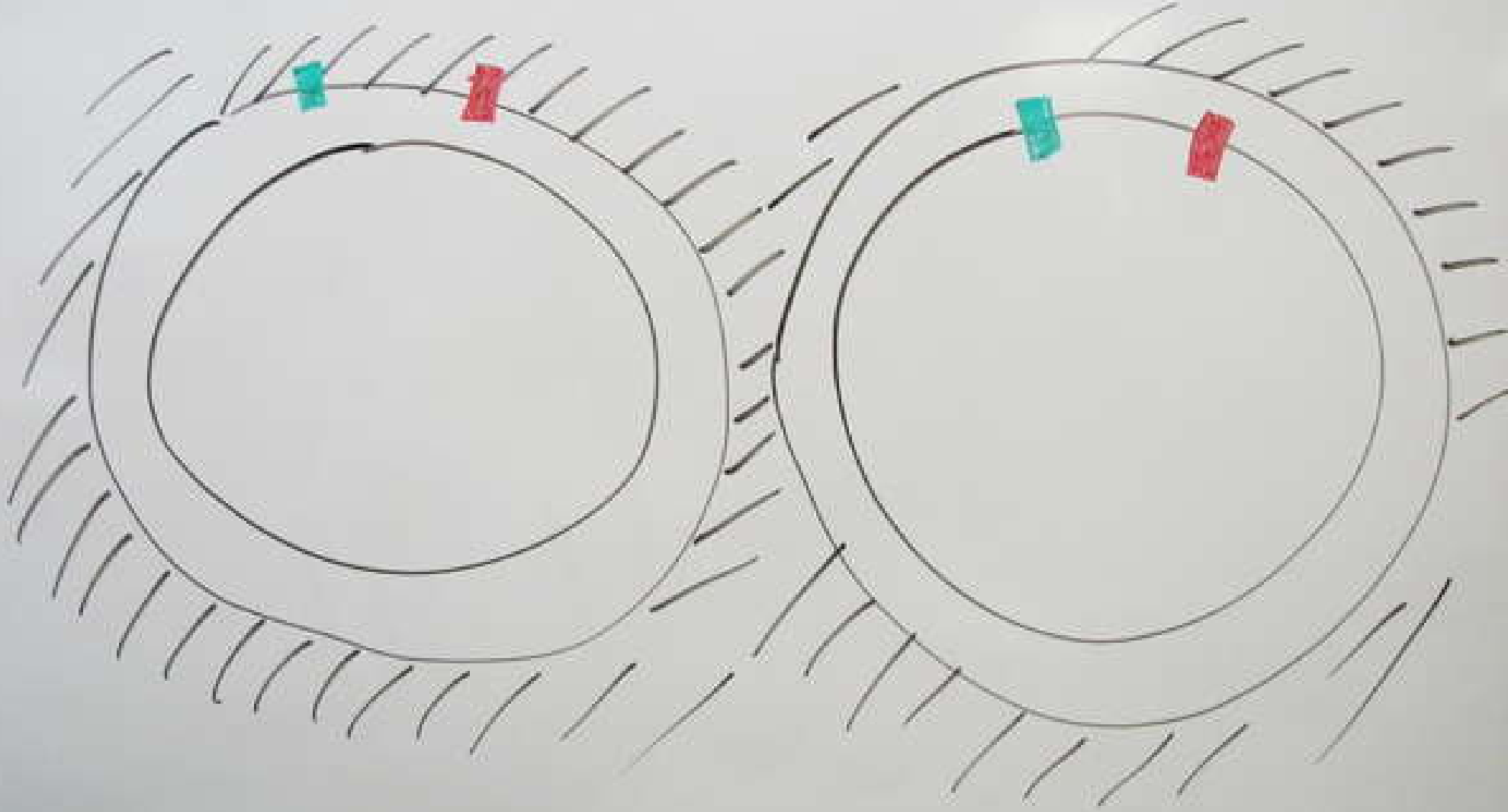

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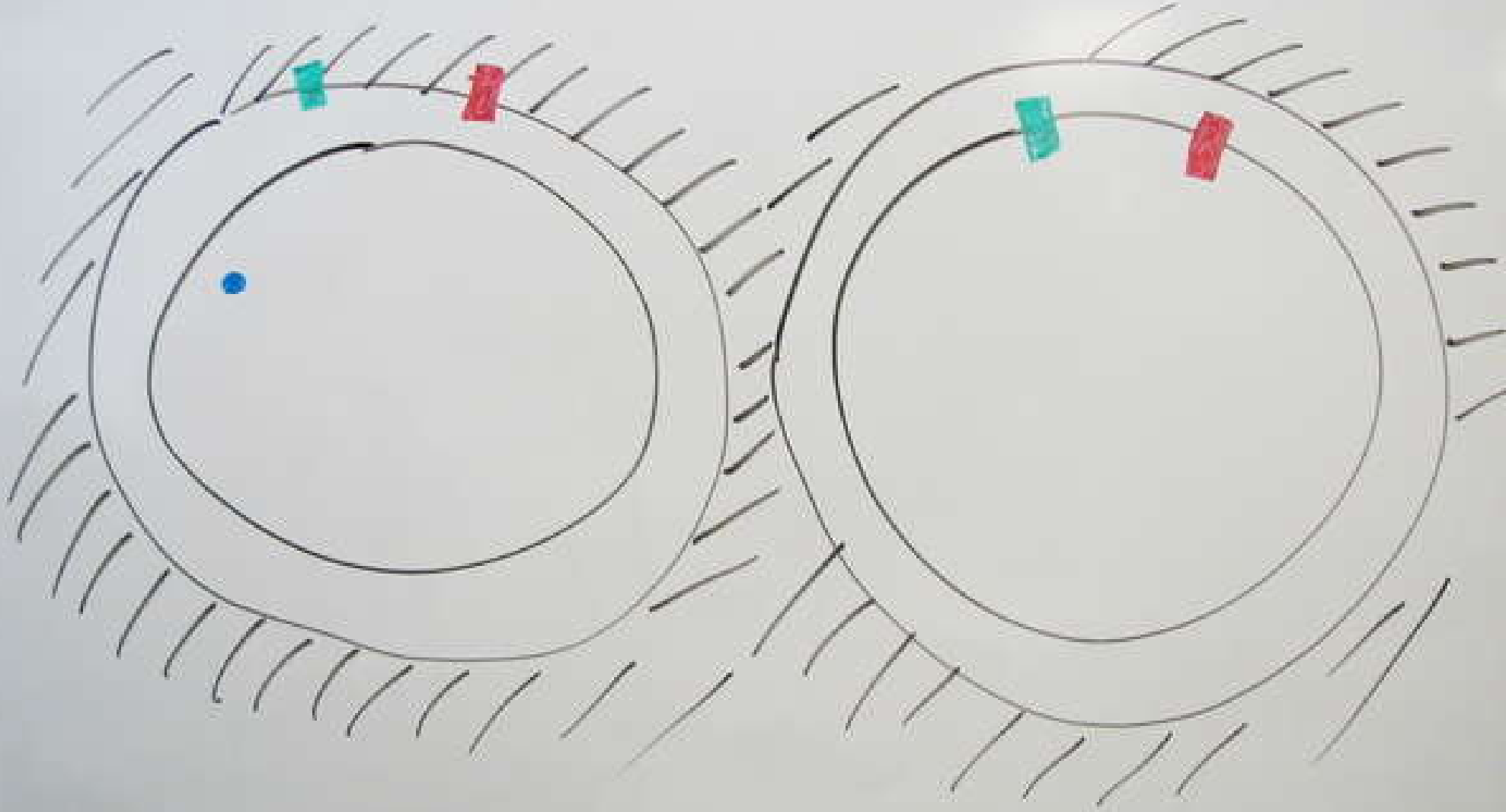

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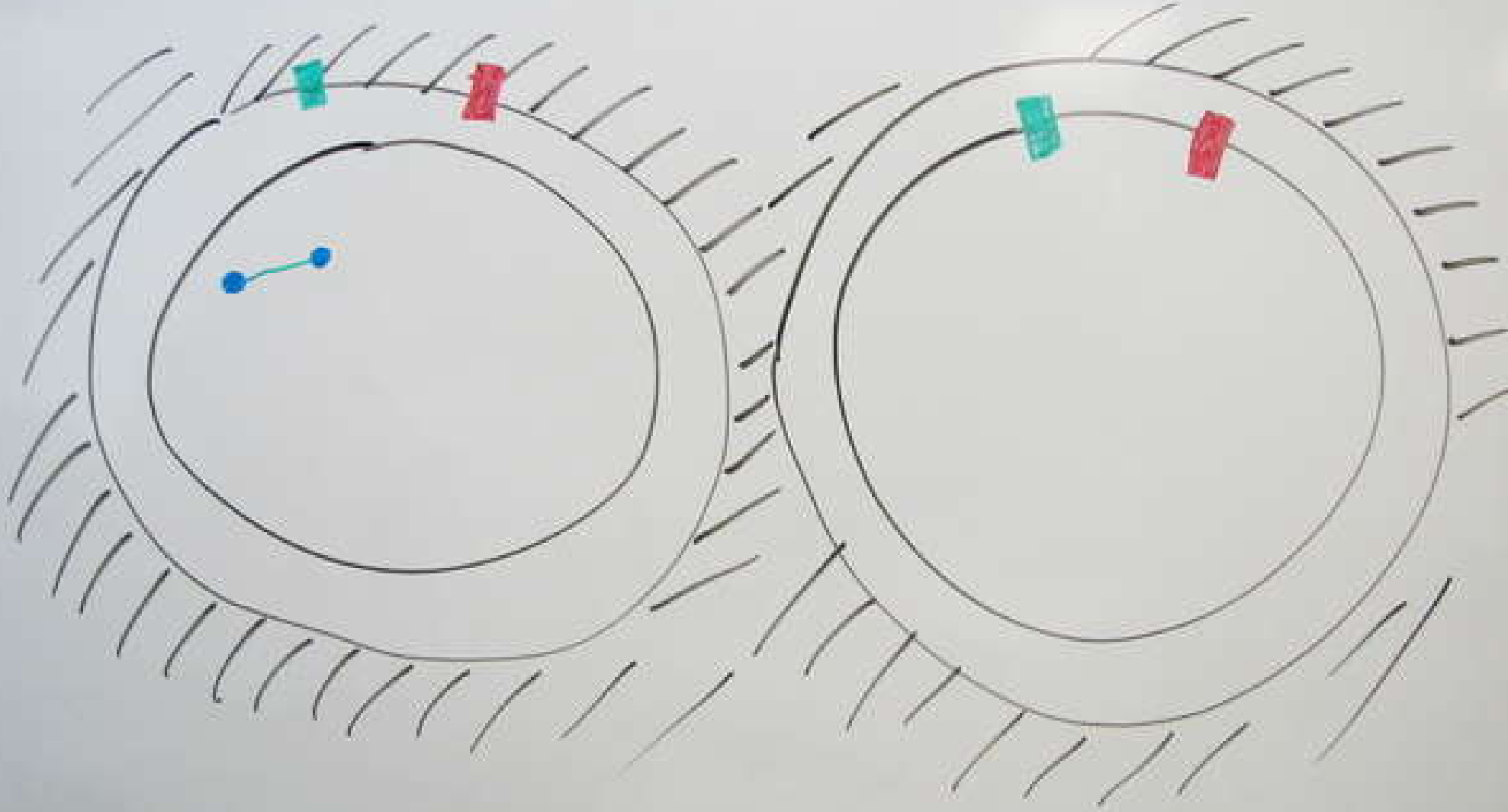

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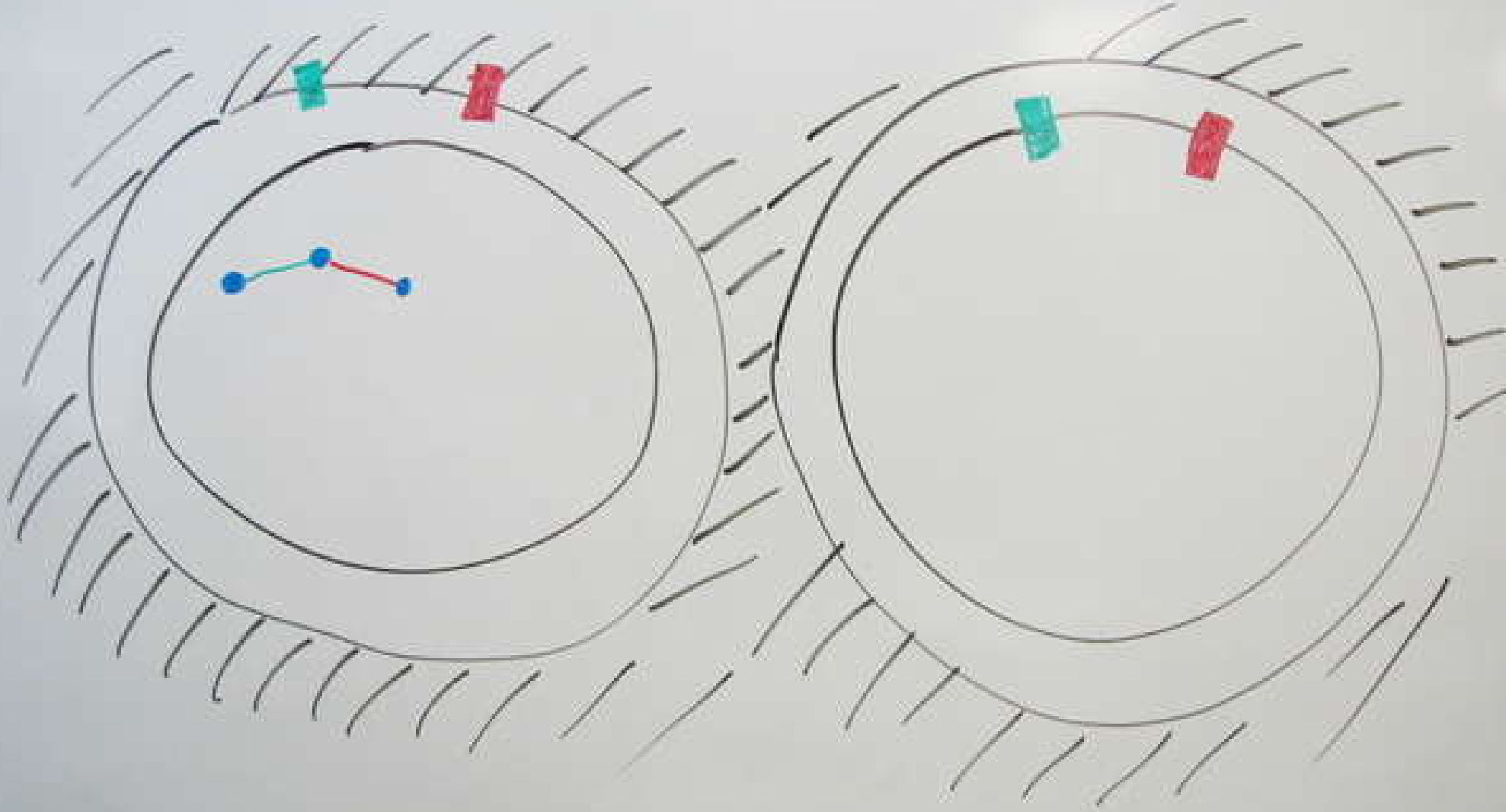

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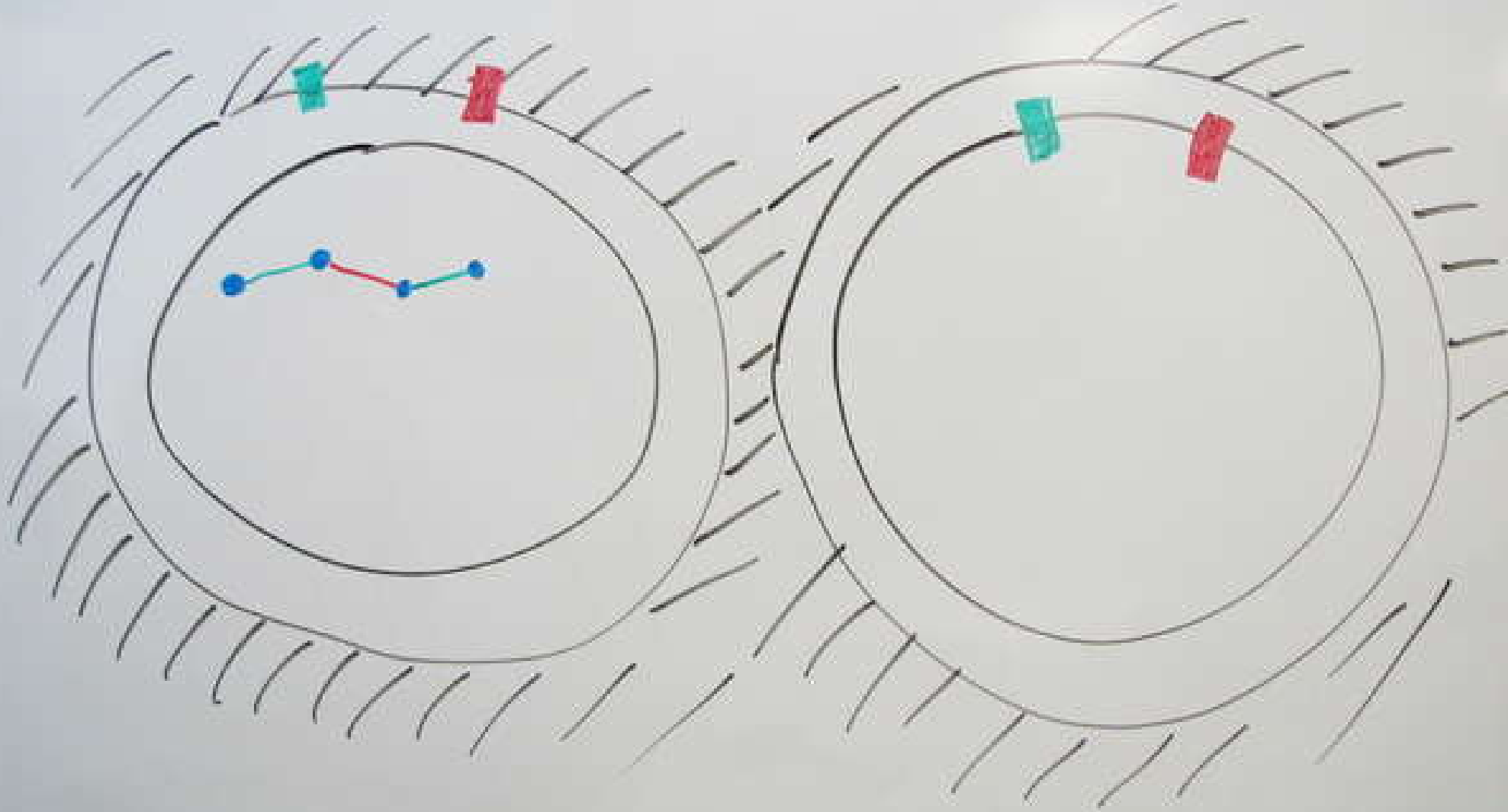

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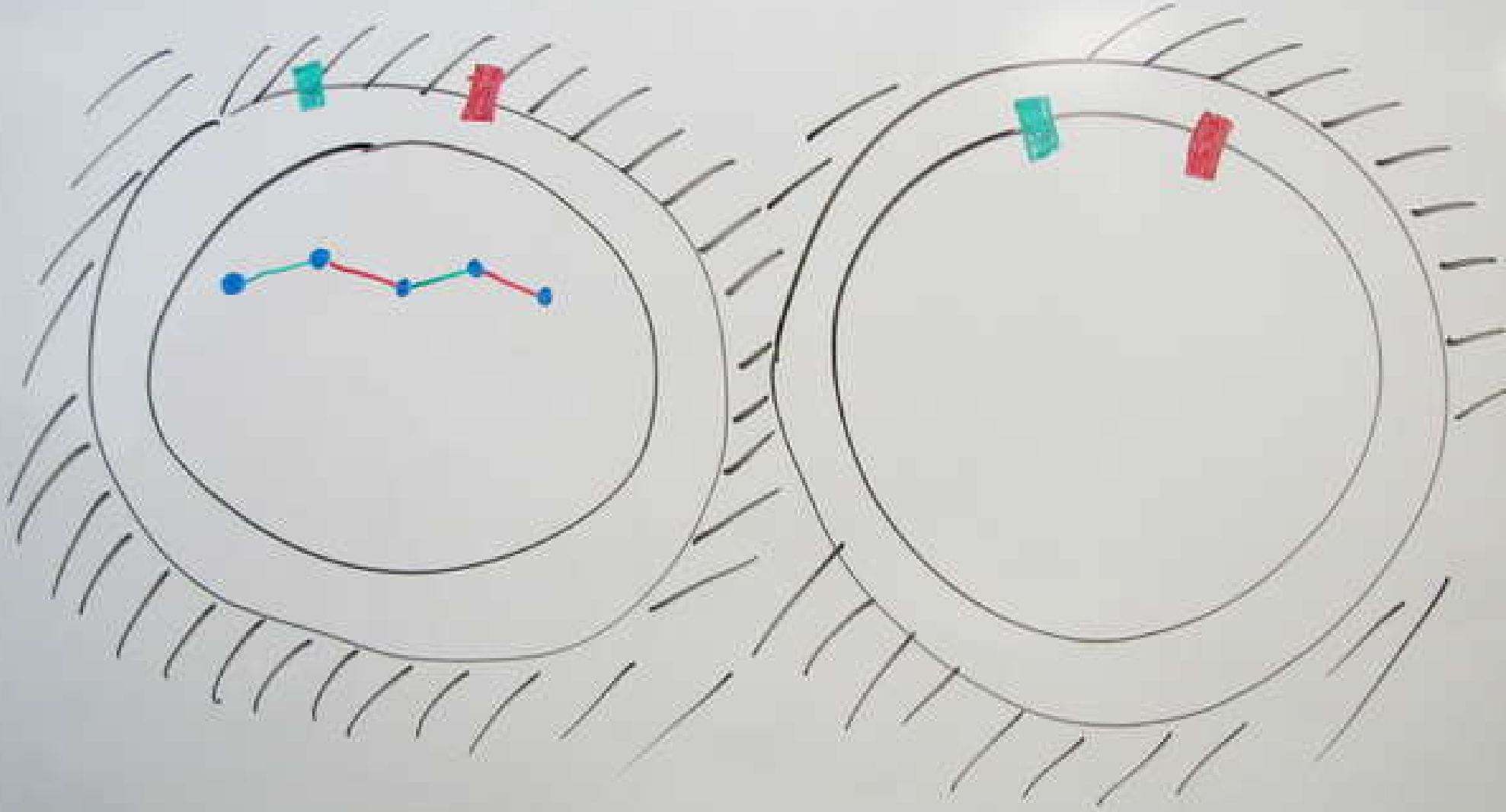

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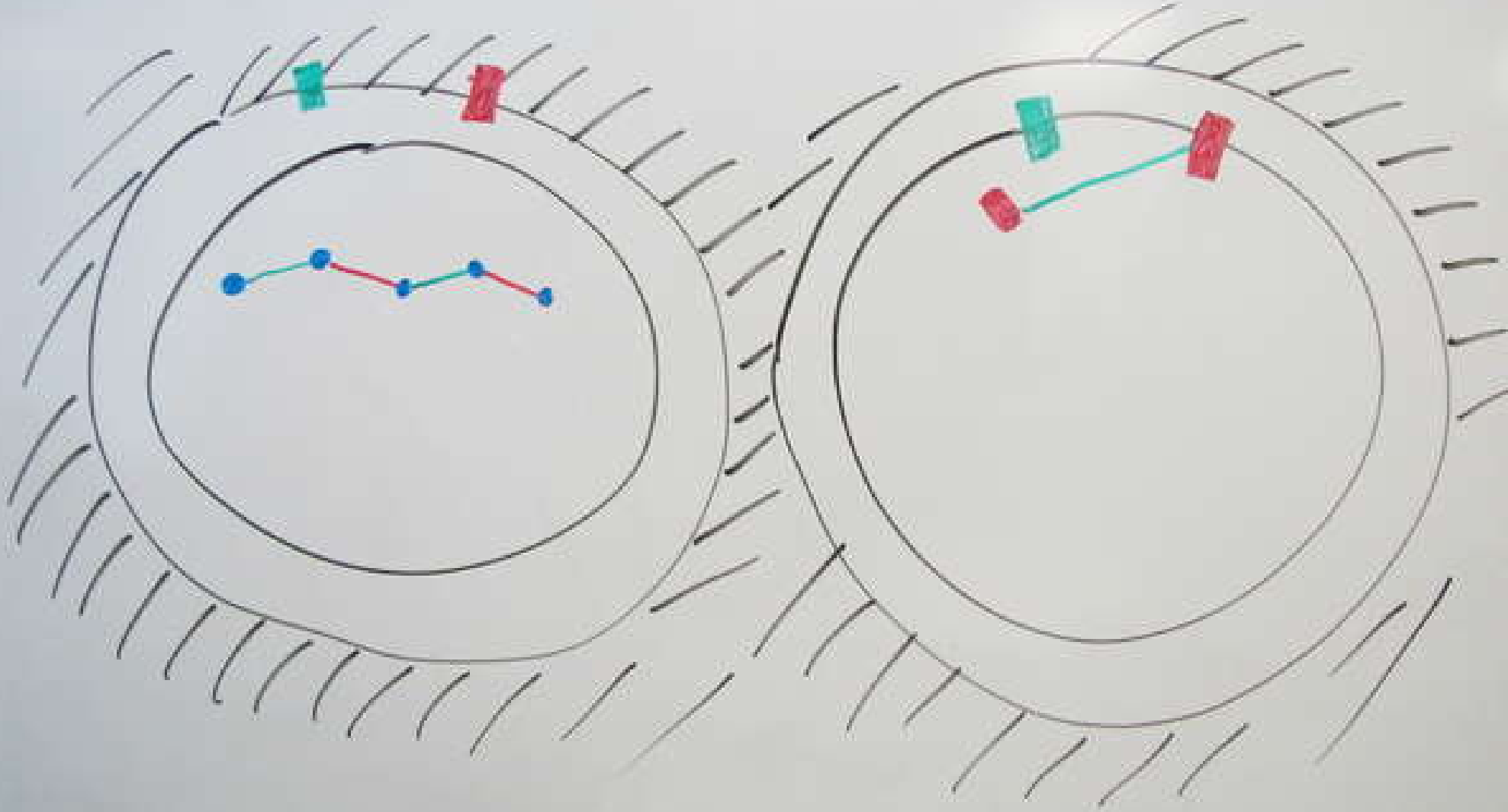

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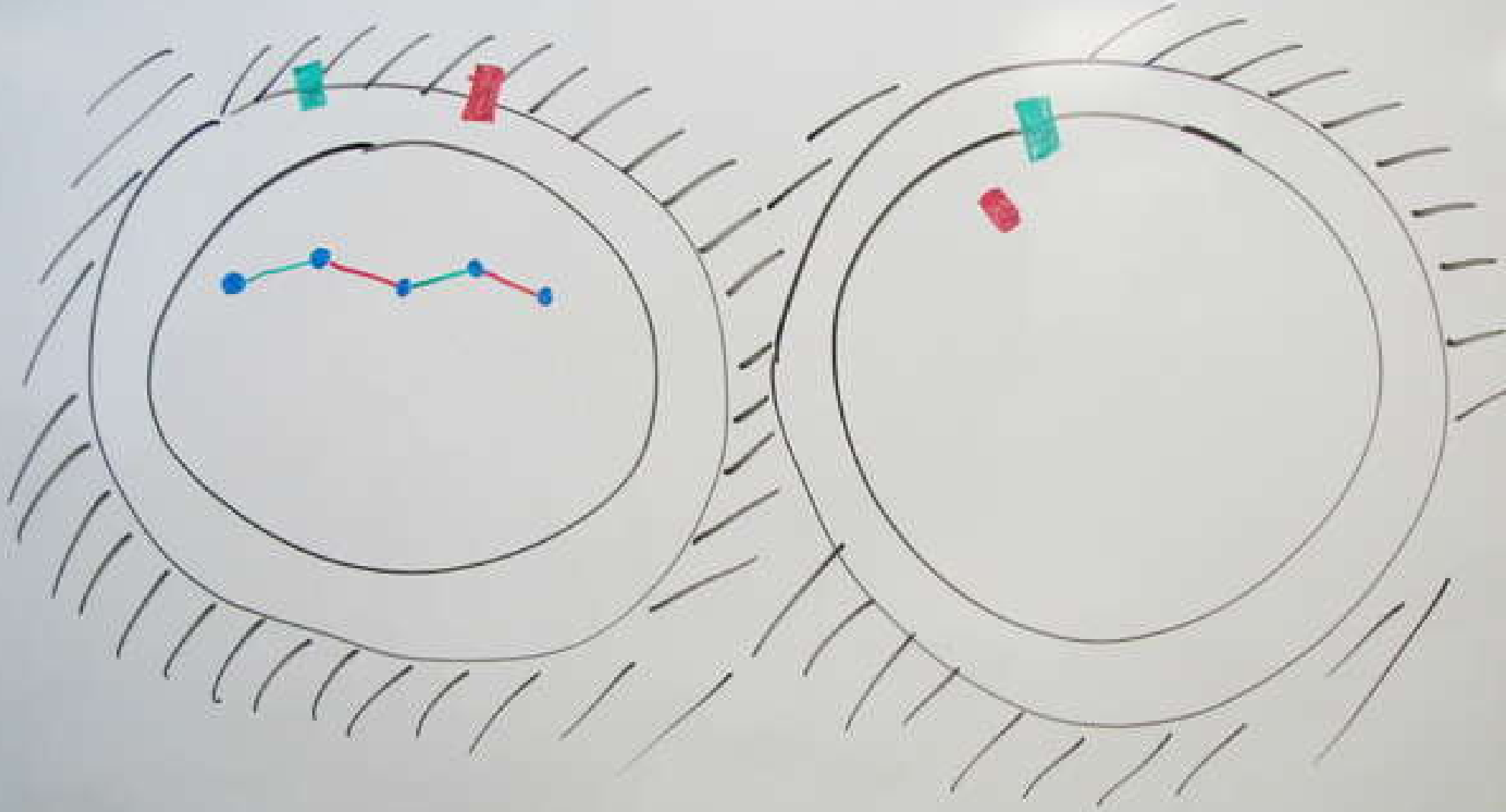

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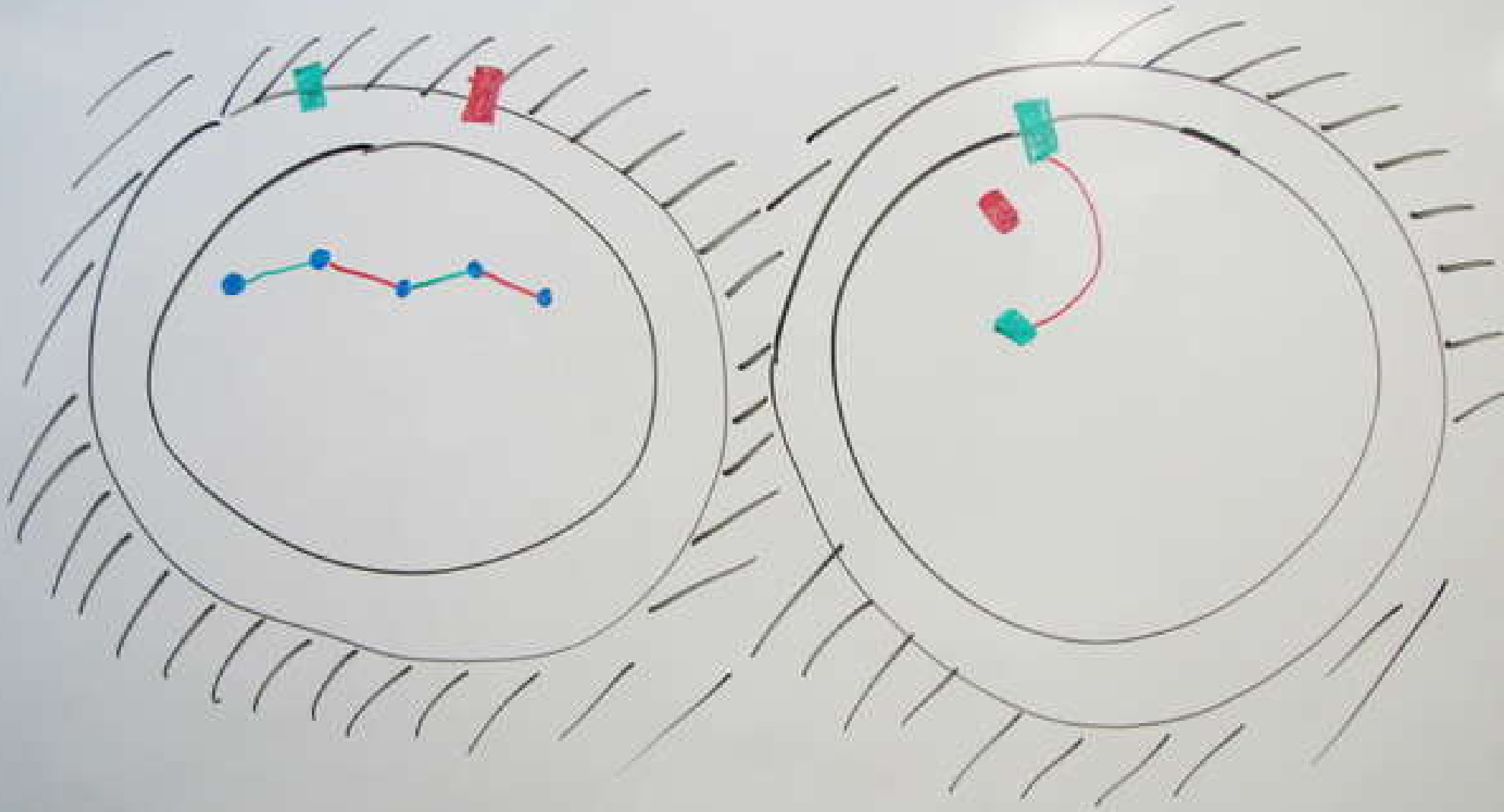

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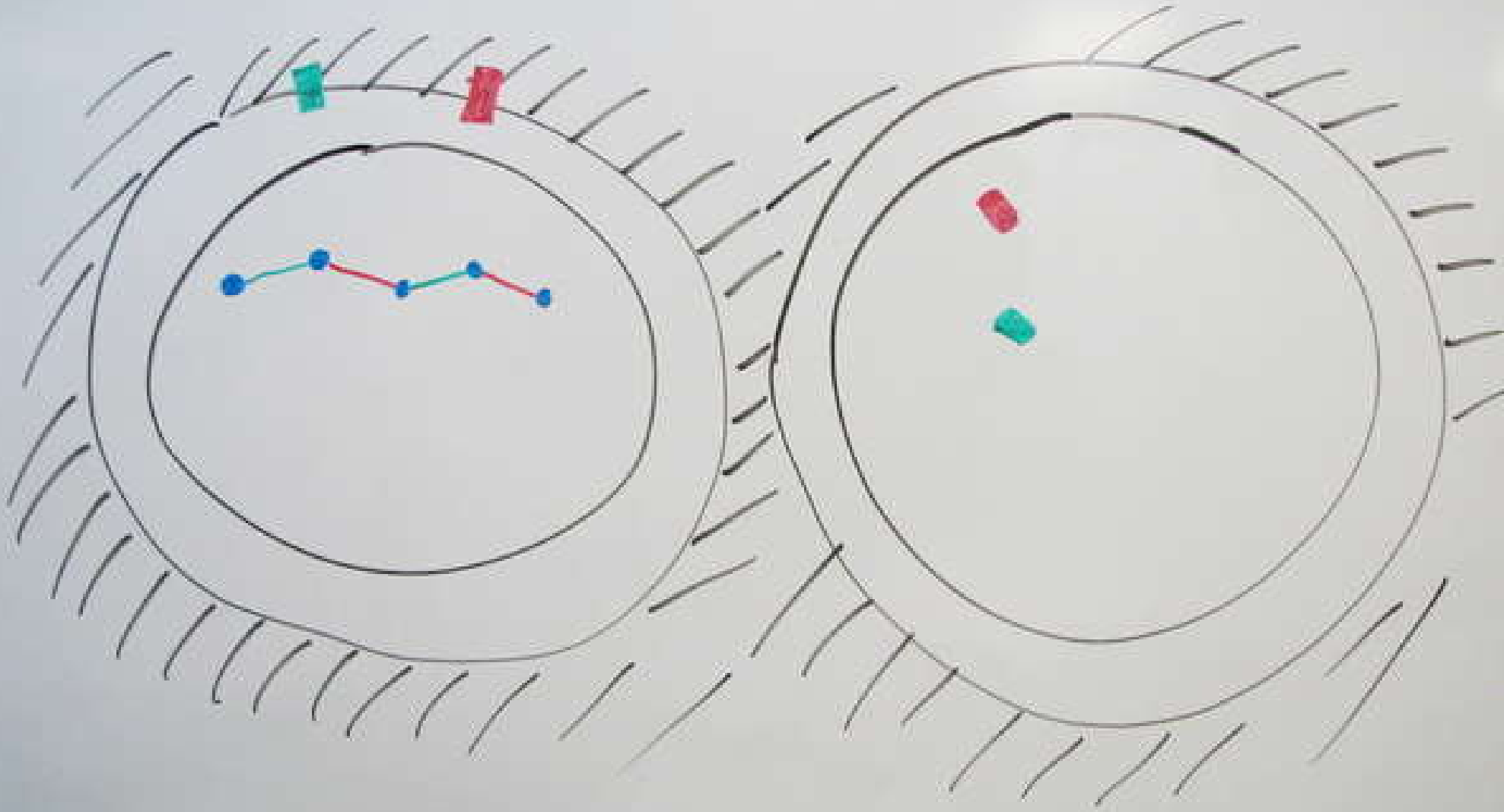

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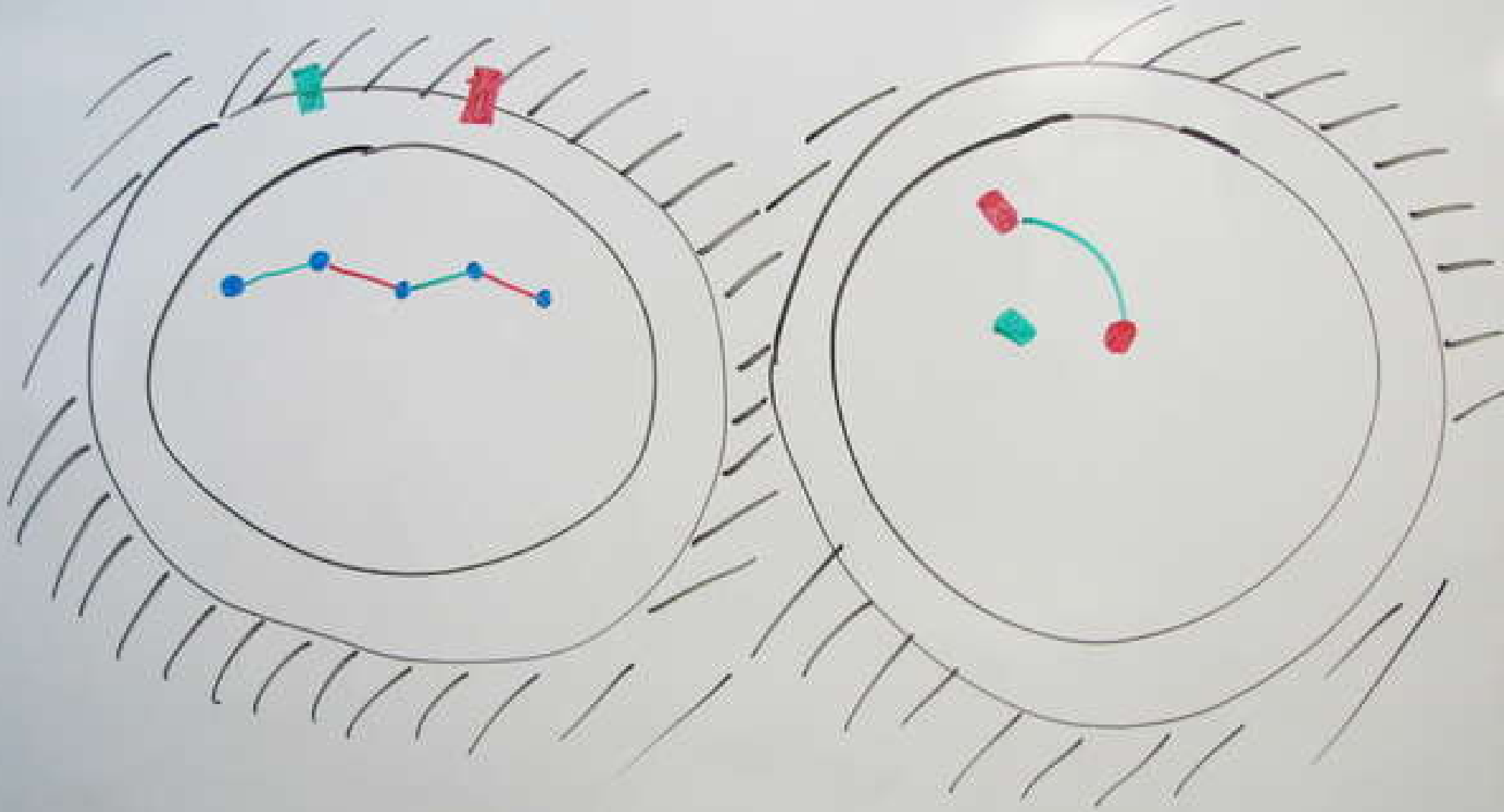

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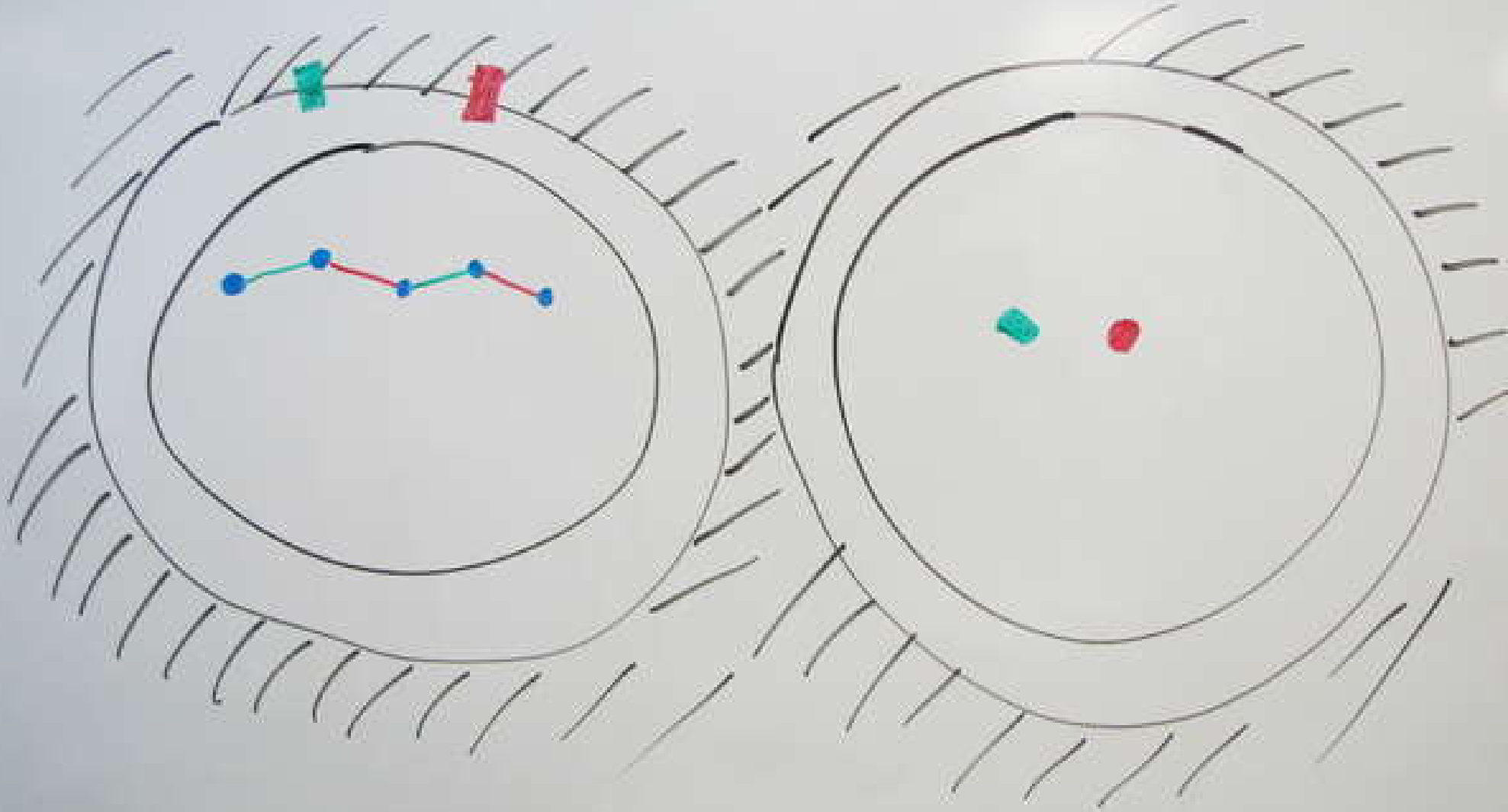

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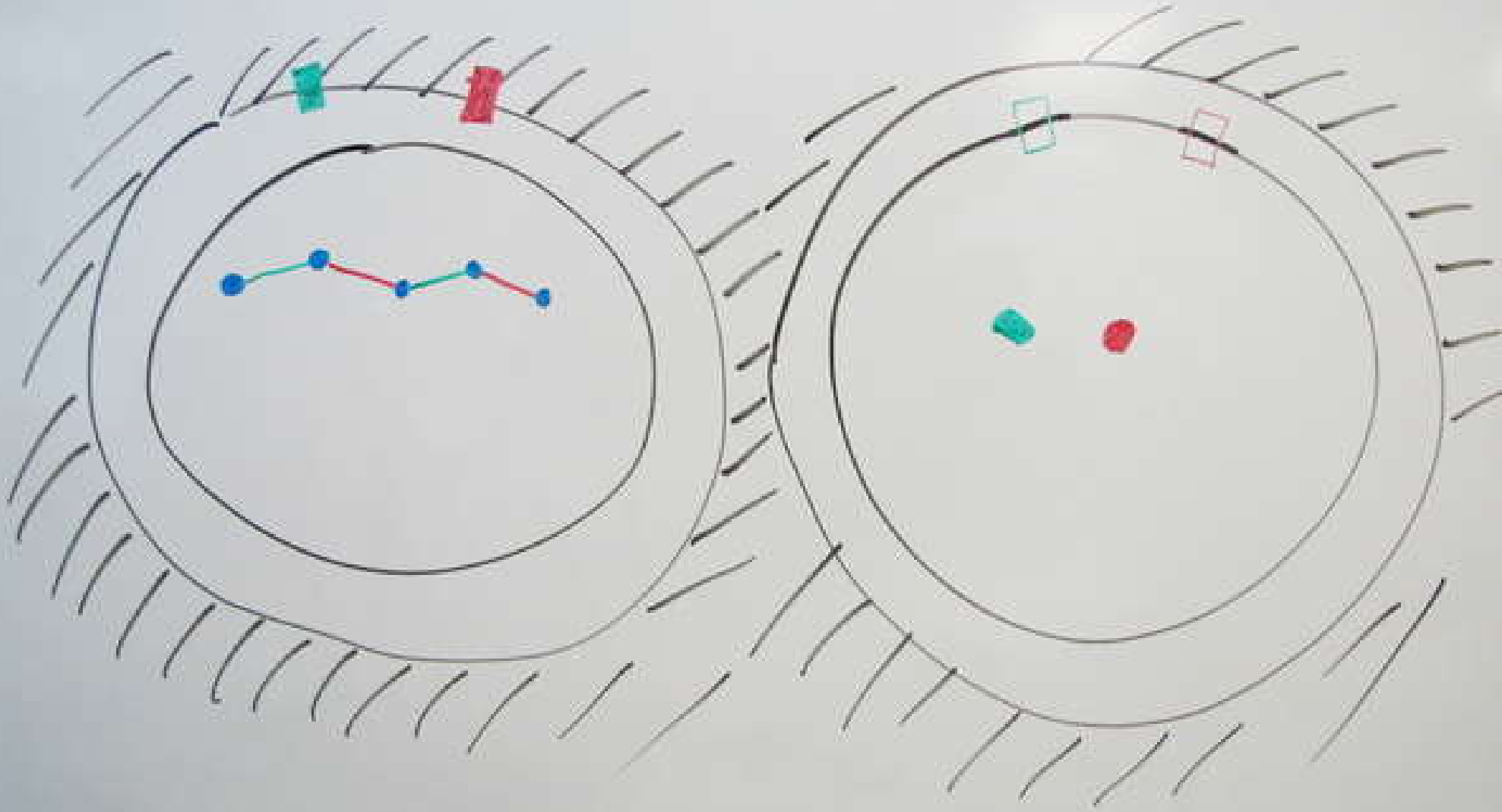

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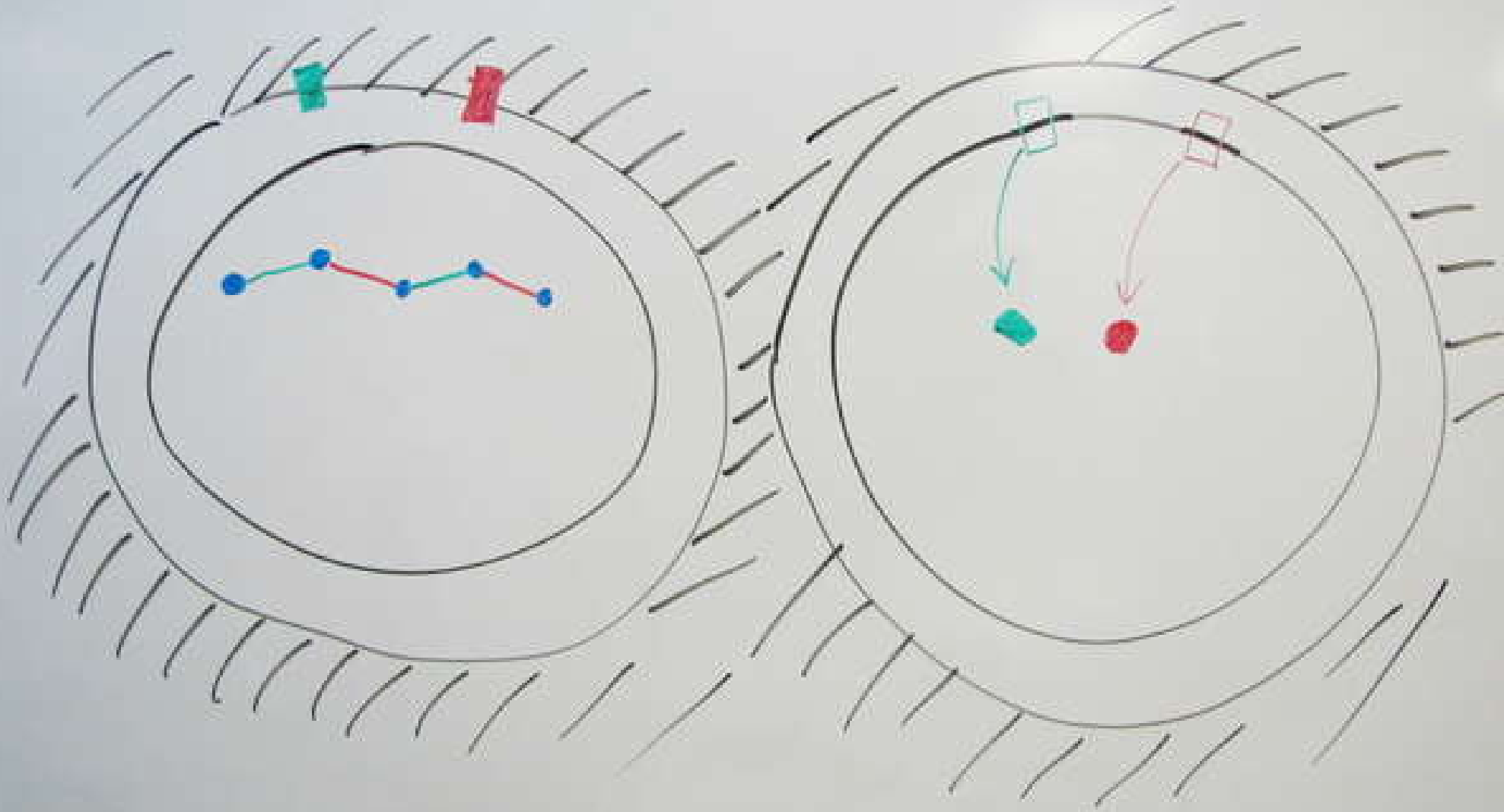

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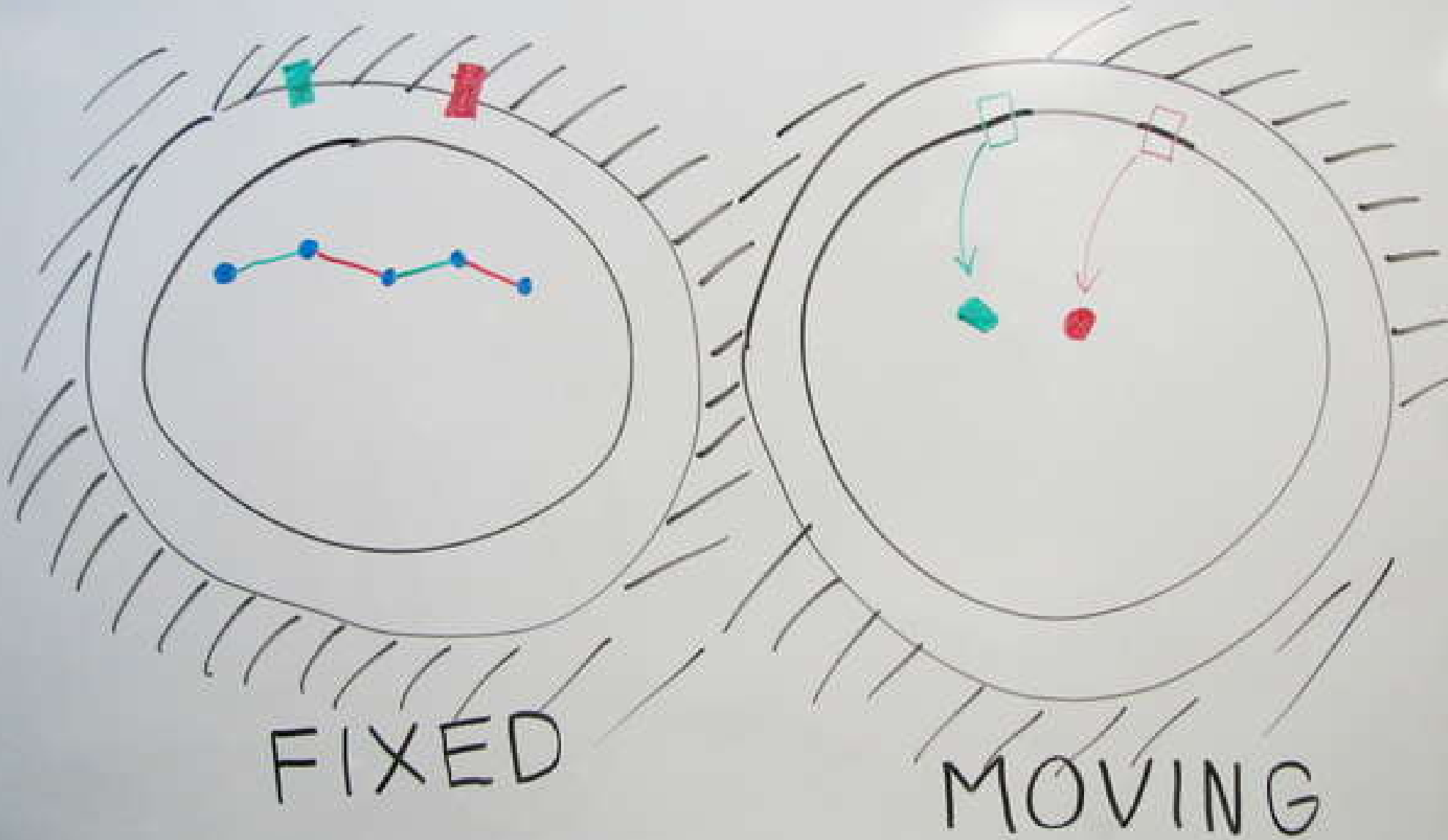

Rotation Angles

Rotation Angles

Rational / Irrational

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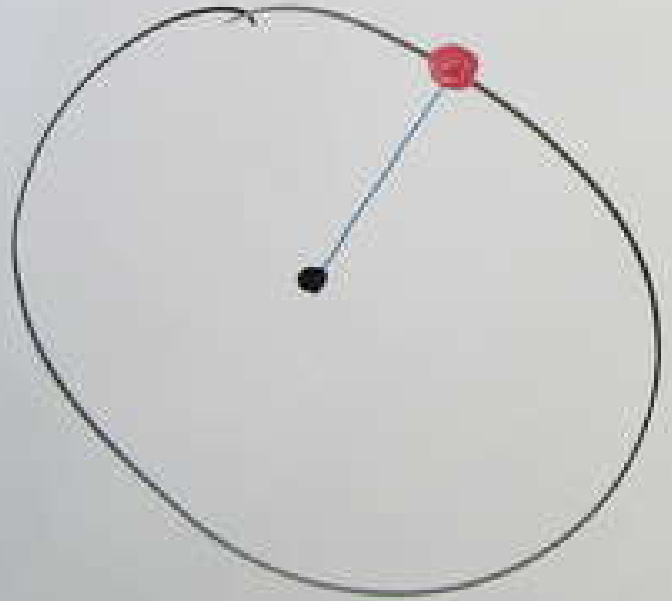

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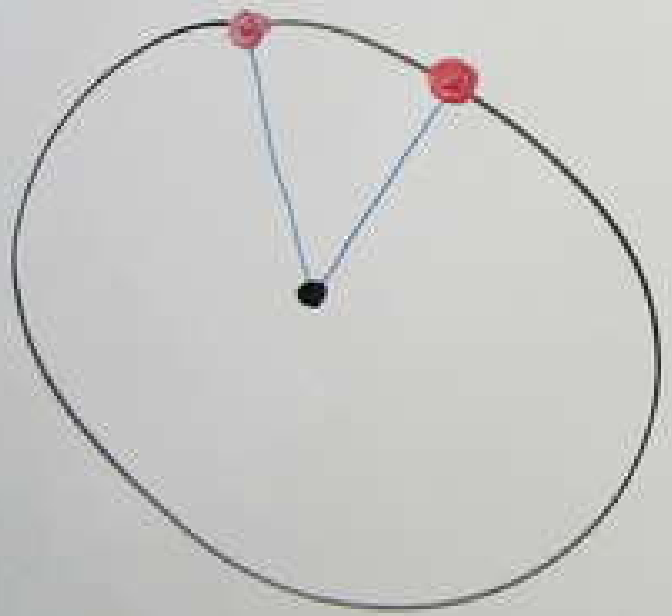

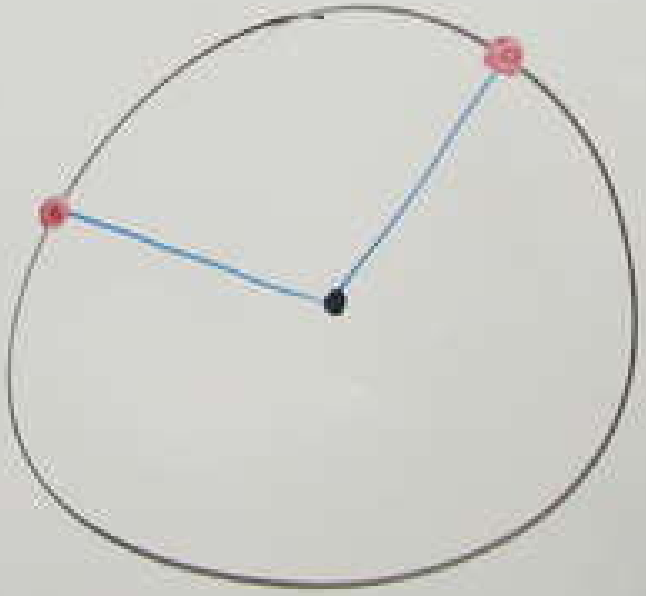

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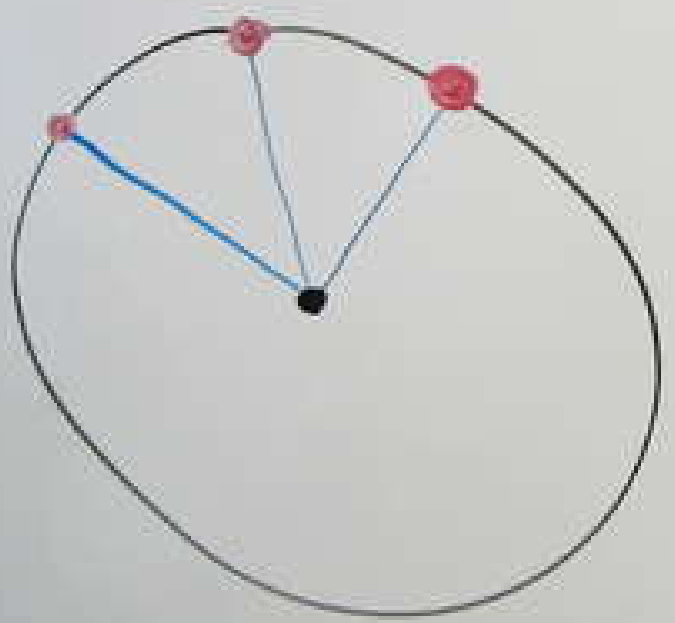

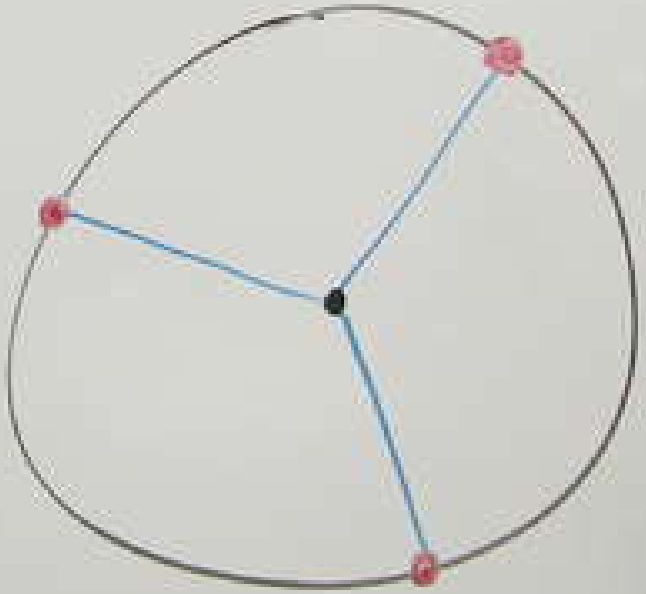

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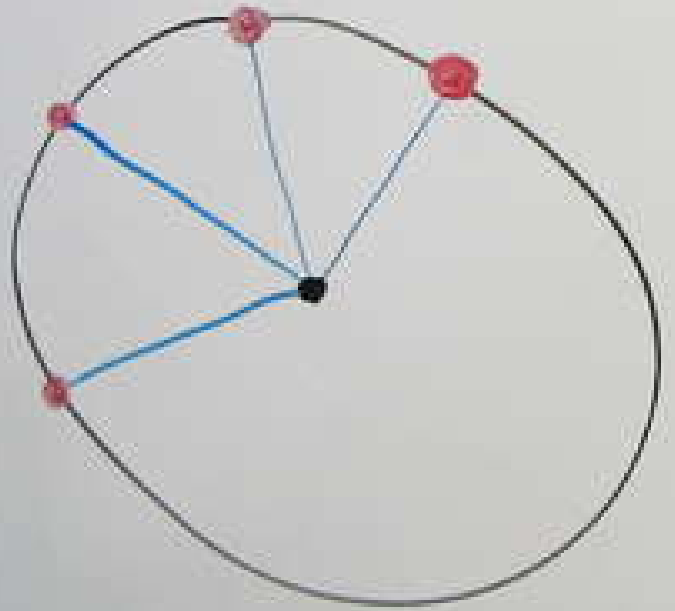

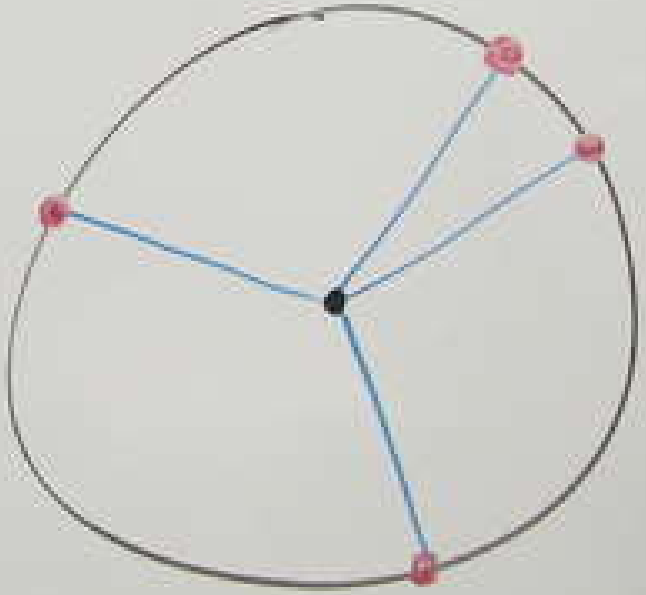

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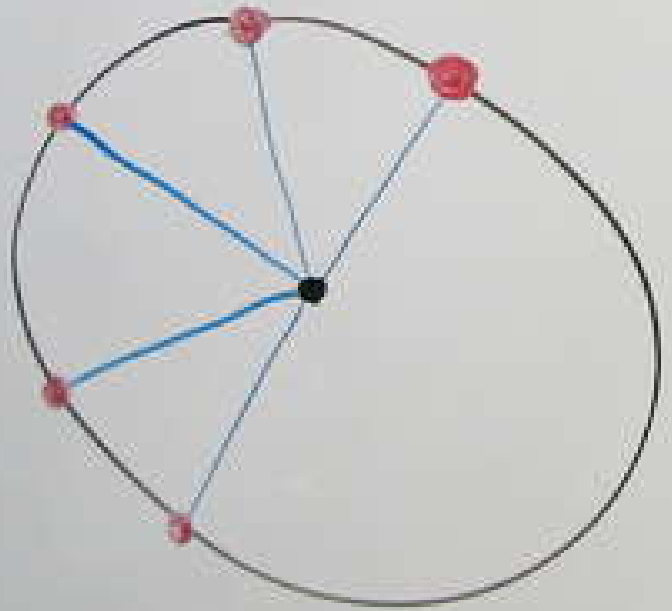

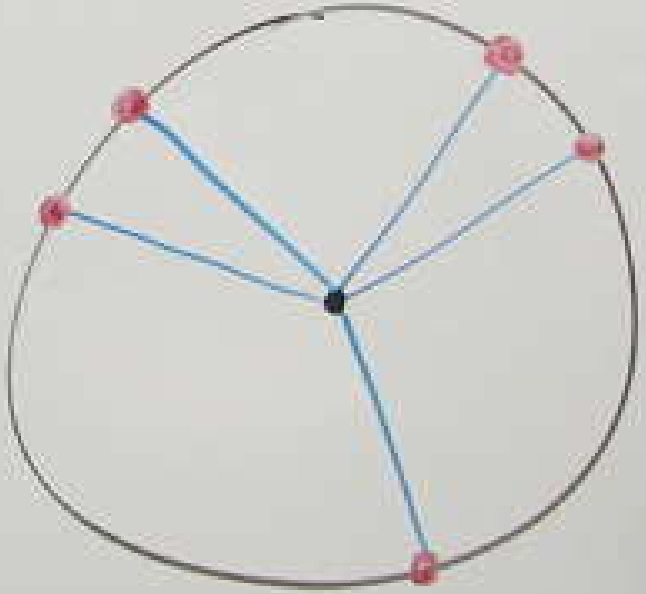

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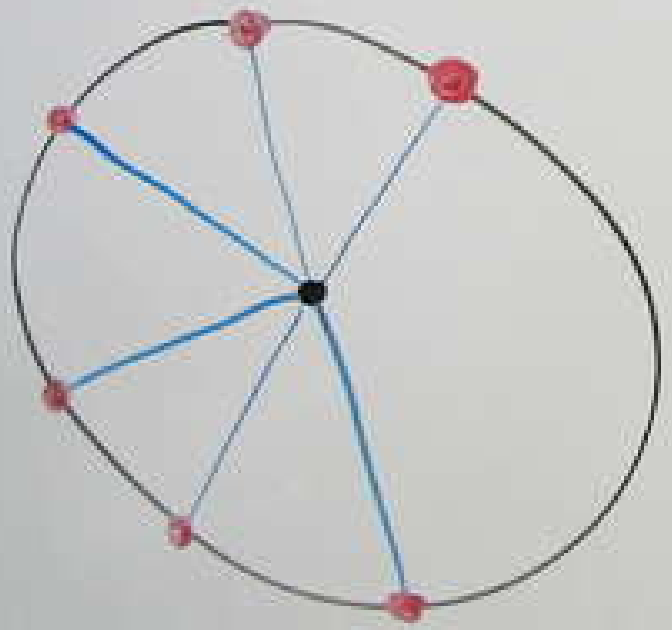

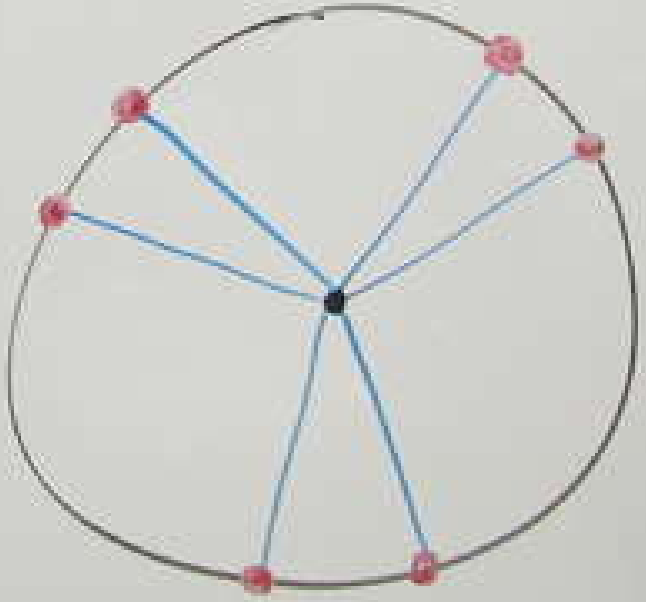

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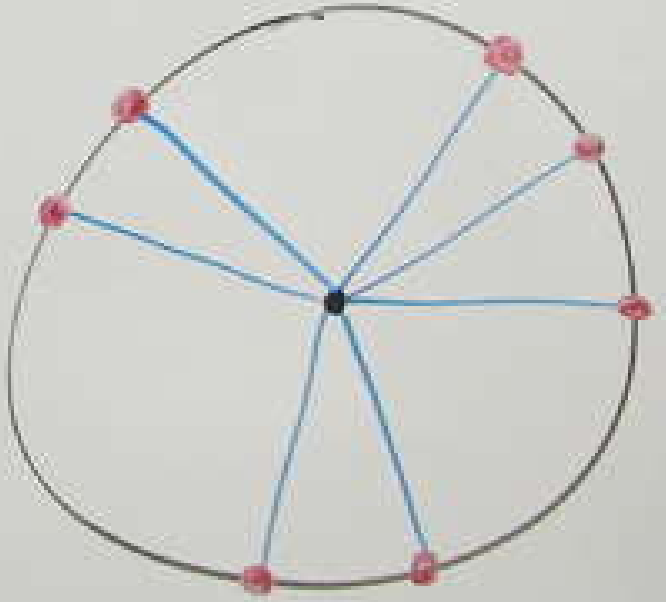

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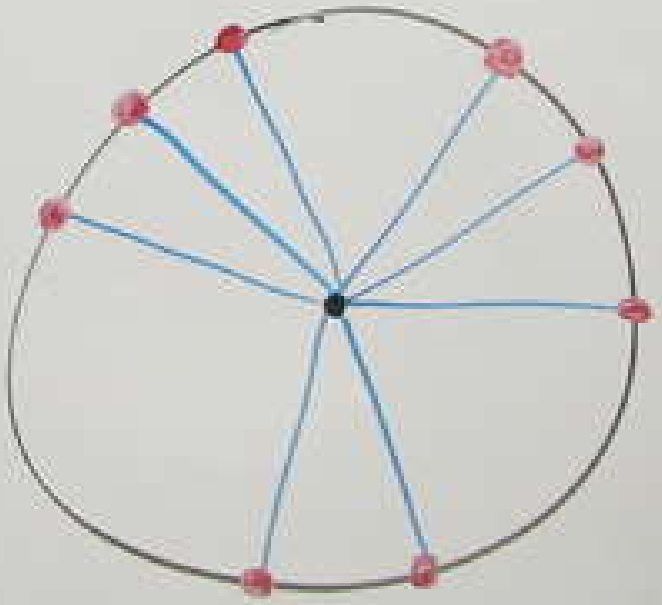

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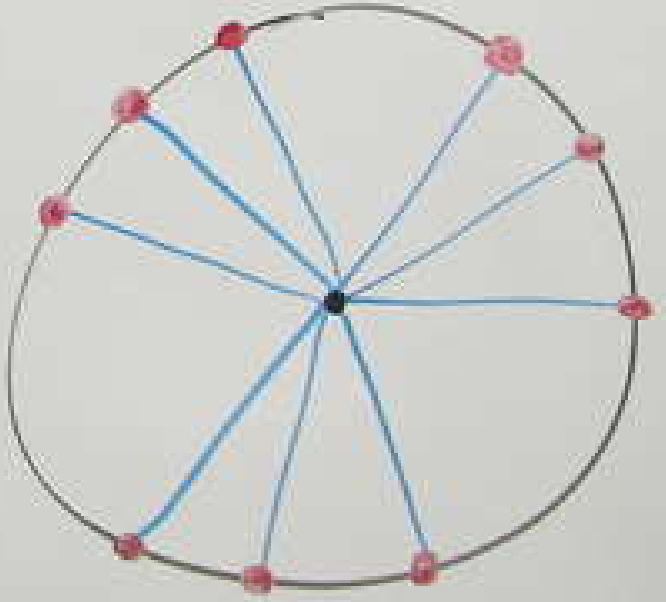

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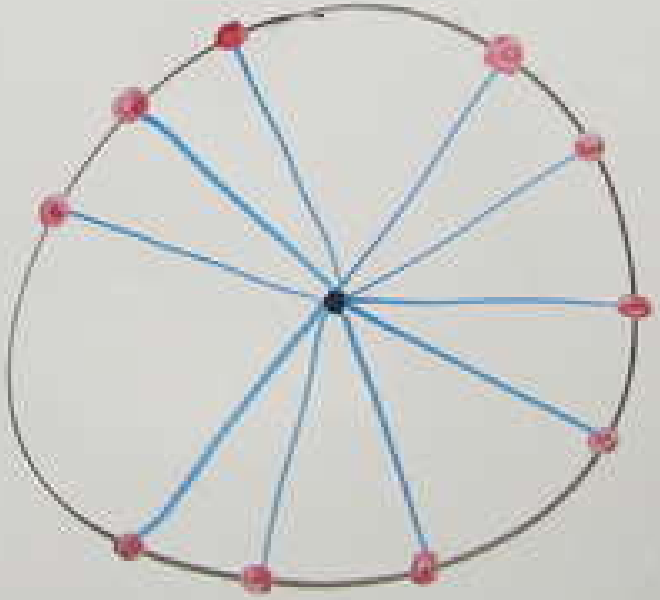

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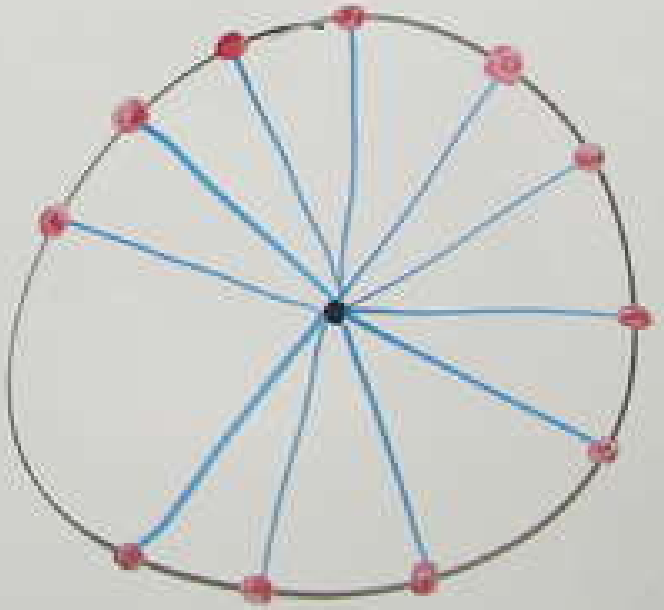

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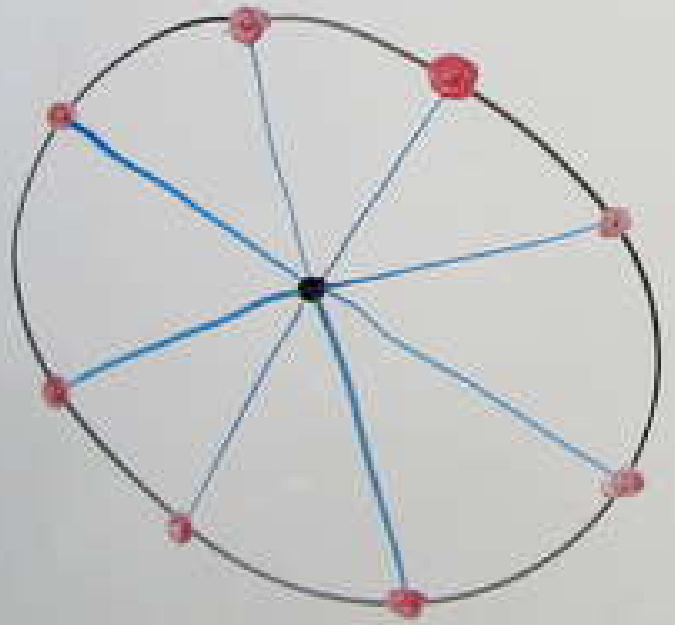

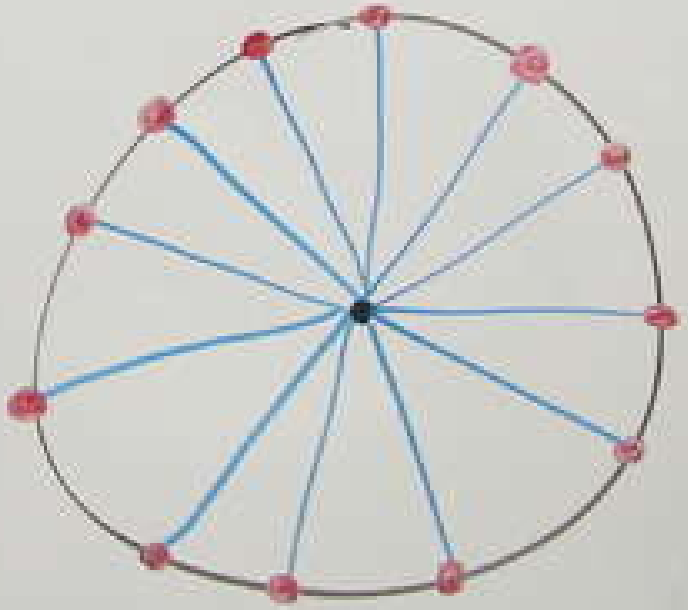

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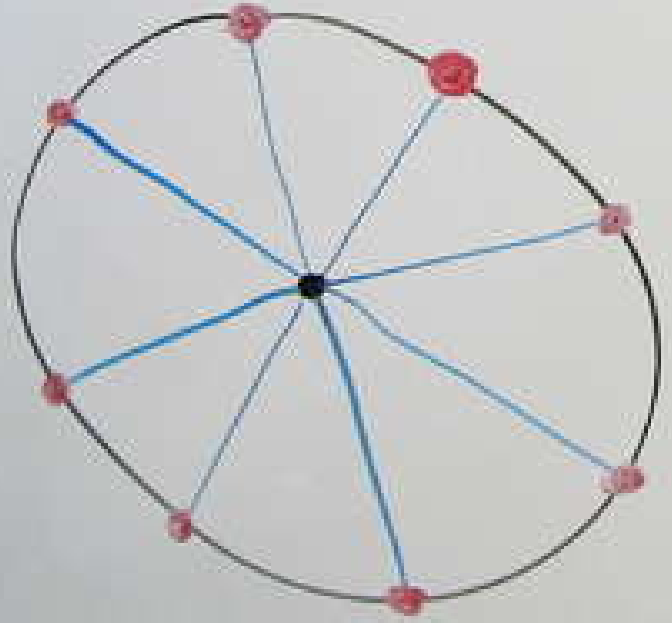

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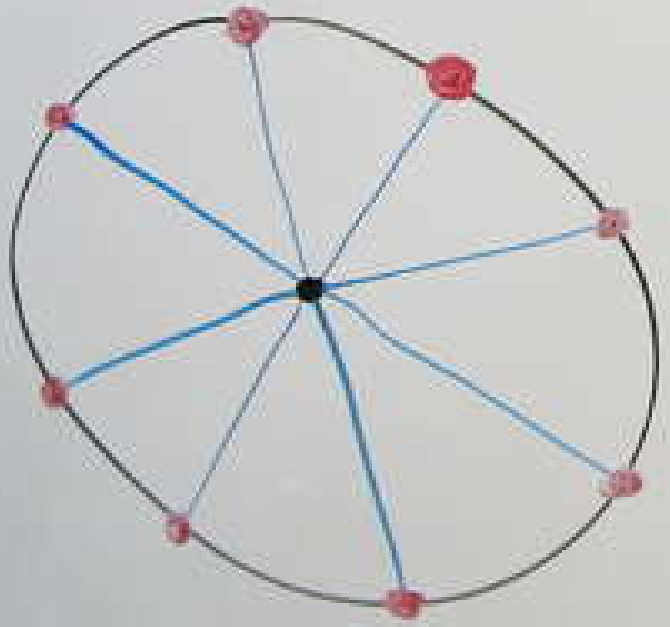
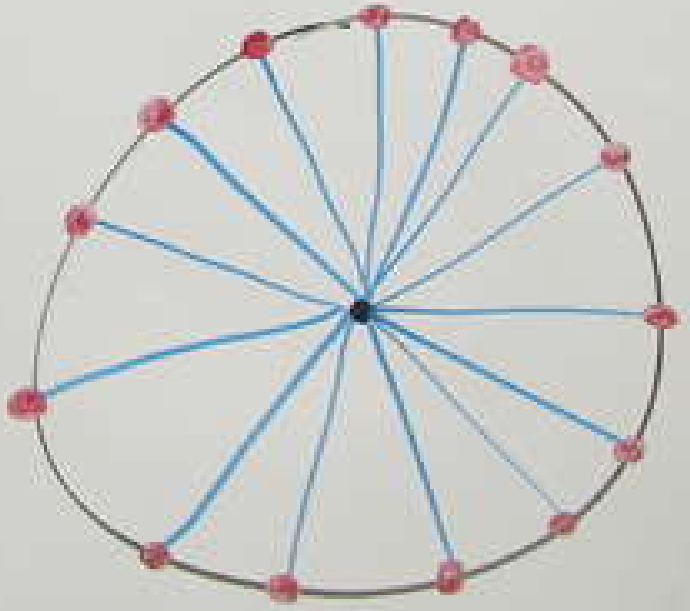
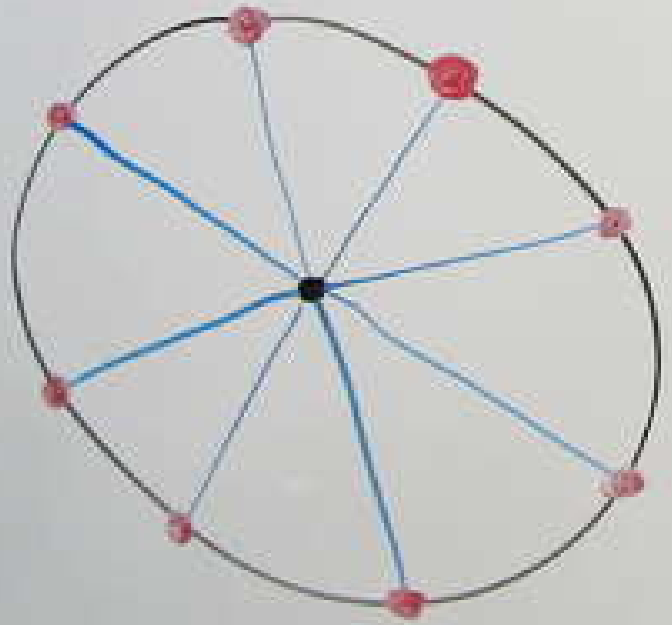
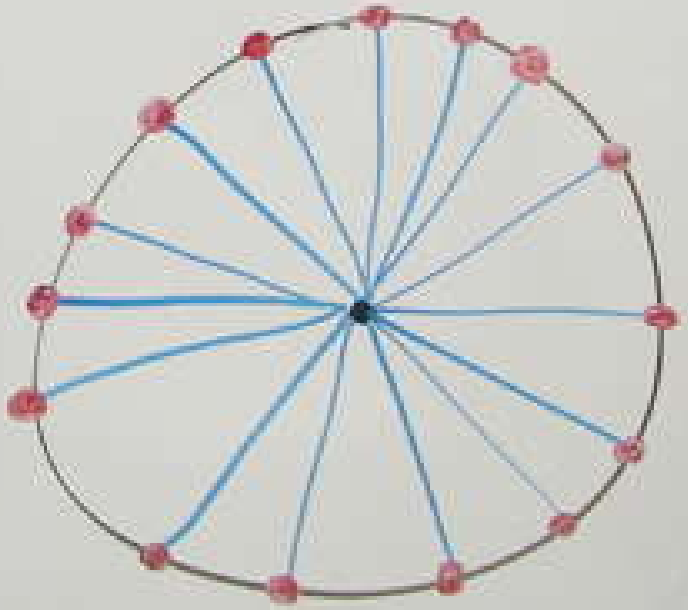




Rotation Angles

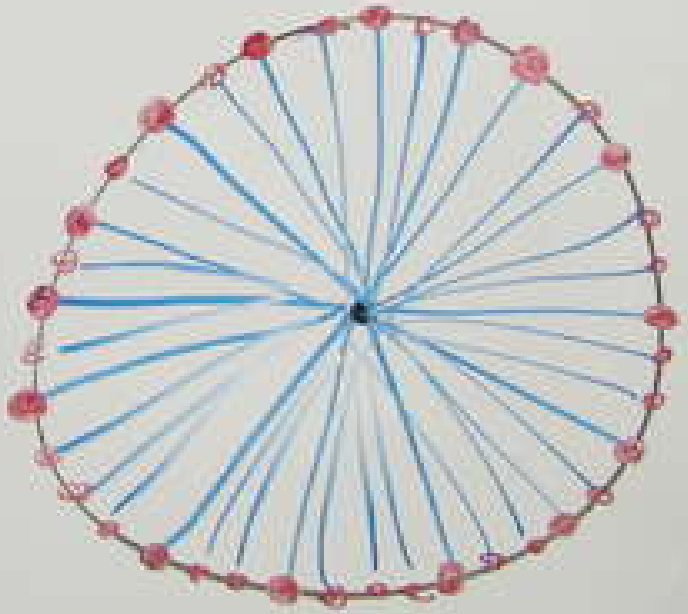
Rational / Irrational





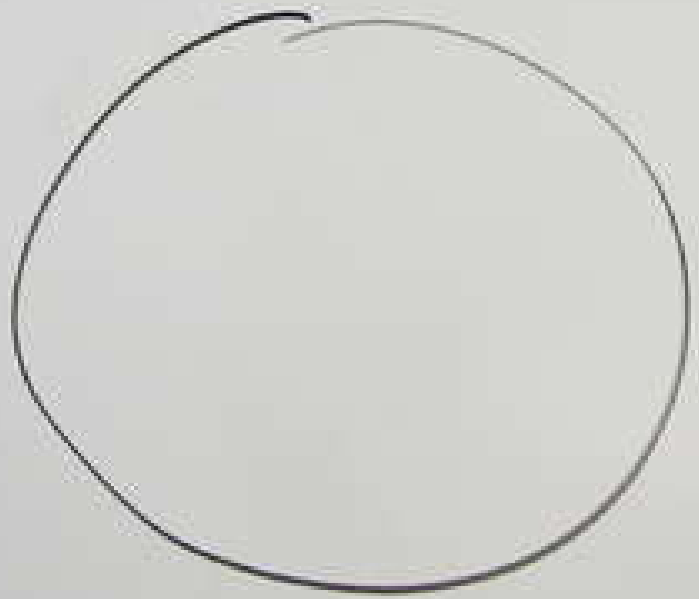
Rotation Angles

Rational / Irrational



Rotation Angles

Rational / Irrational



Rotation Angles

Rational / Irrational

Rotation Angles

Rational / Irrational

Rotation Angles

Rational / Irrational

Rotation Angles

Rational / Irrational

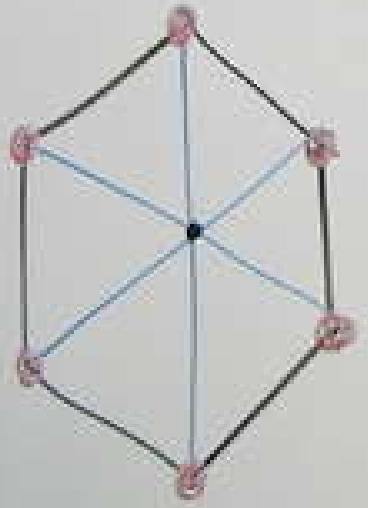
Rotation Angles

Rational / Irrational

Rotation Angles

Rational / Irrational

Discrete / Continuous



Rotation Angles

RATIONAL / IRATIONAL

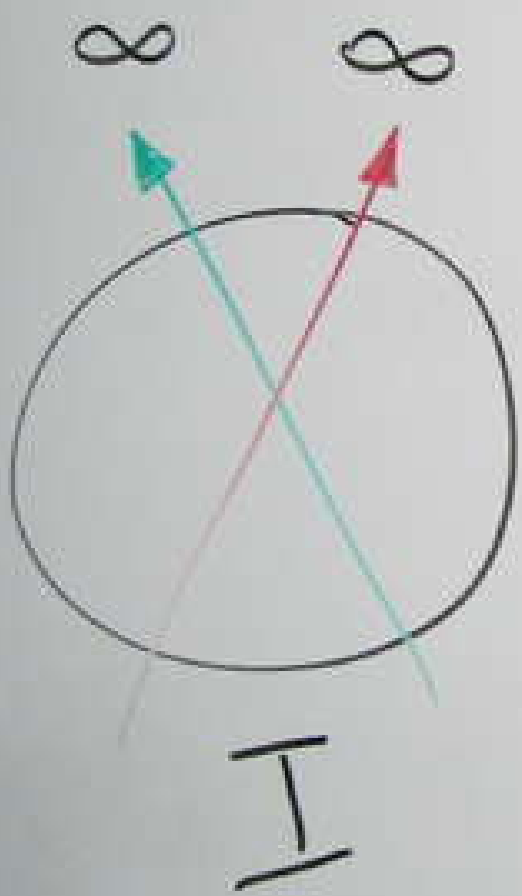
DISCRETE / CONTINUOUS

n

∞

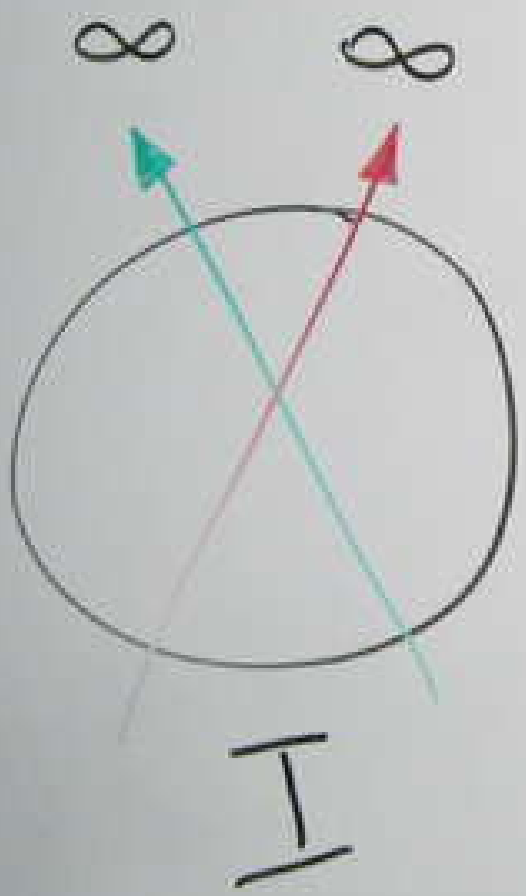
Three Cases

Three Cases

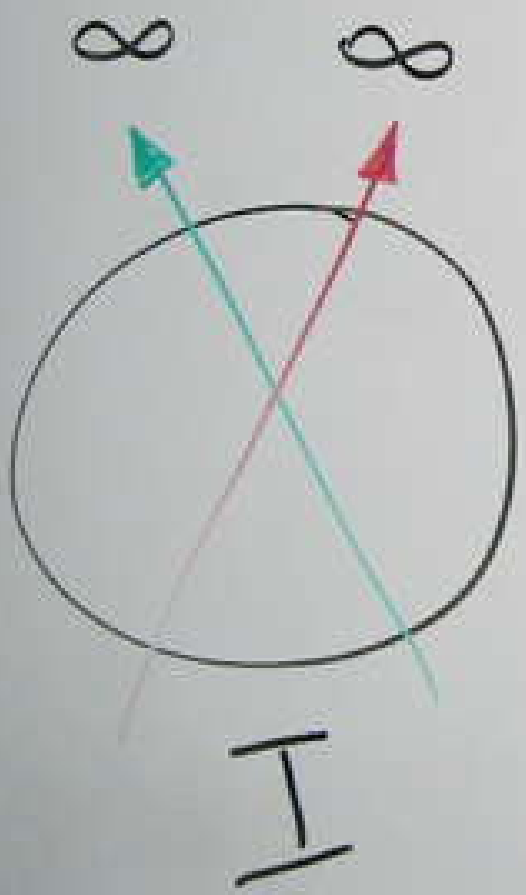


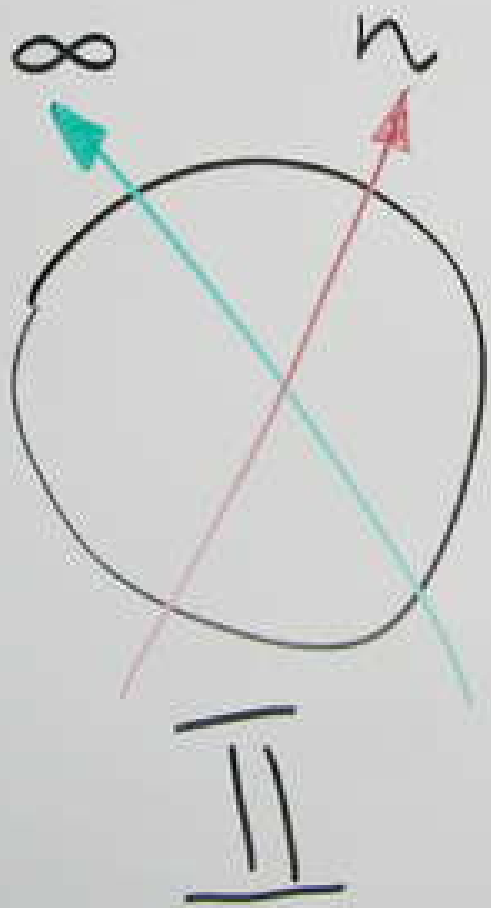
I

Three Cases



Three Cases





Case I (∞, ∞)

CASE I (∞, ∞)

PROOF 1

CASE I (∞, ∞)

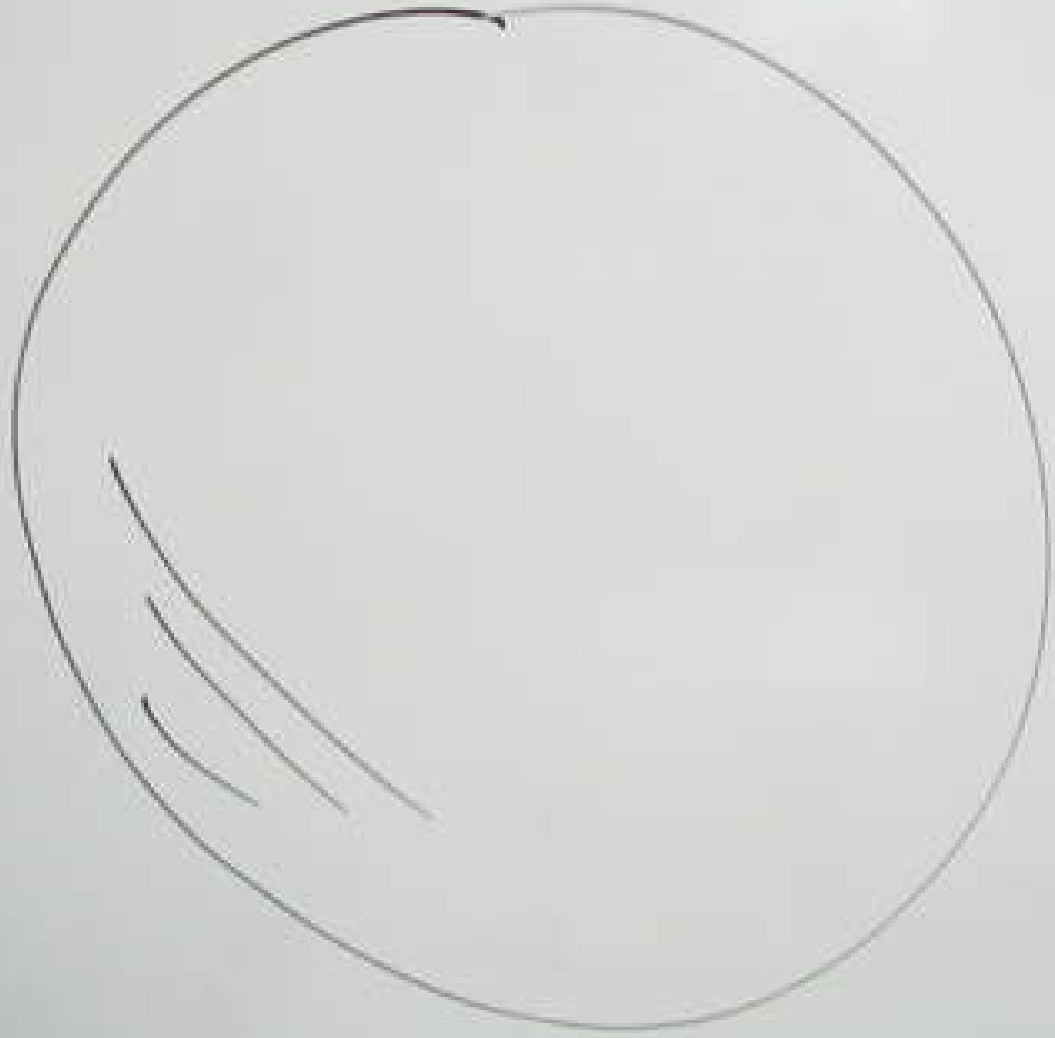
PROOF 1

(moving reference frame)

CASE I (∞, ∞)

PROOF 1

(moving reference frame)

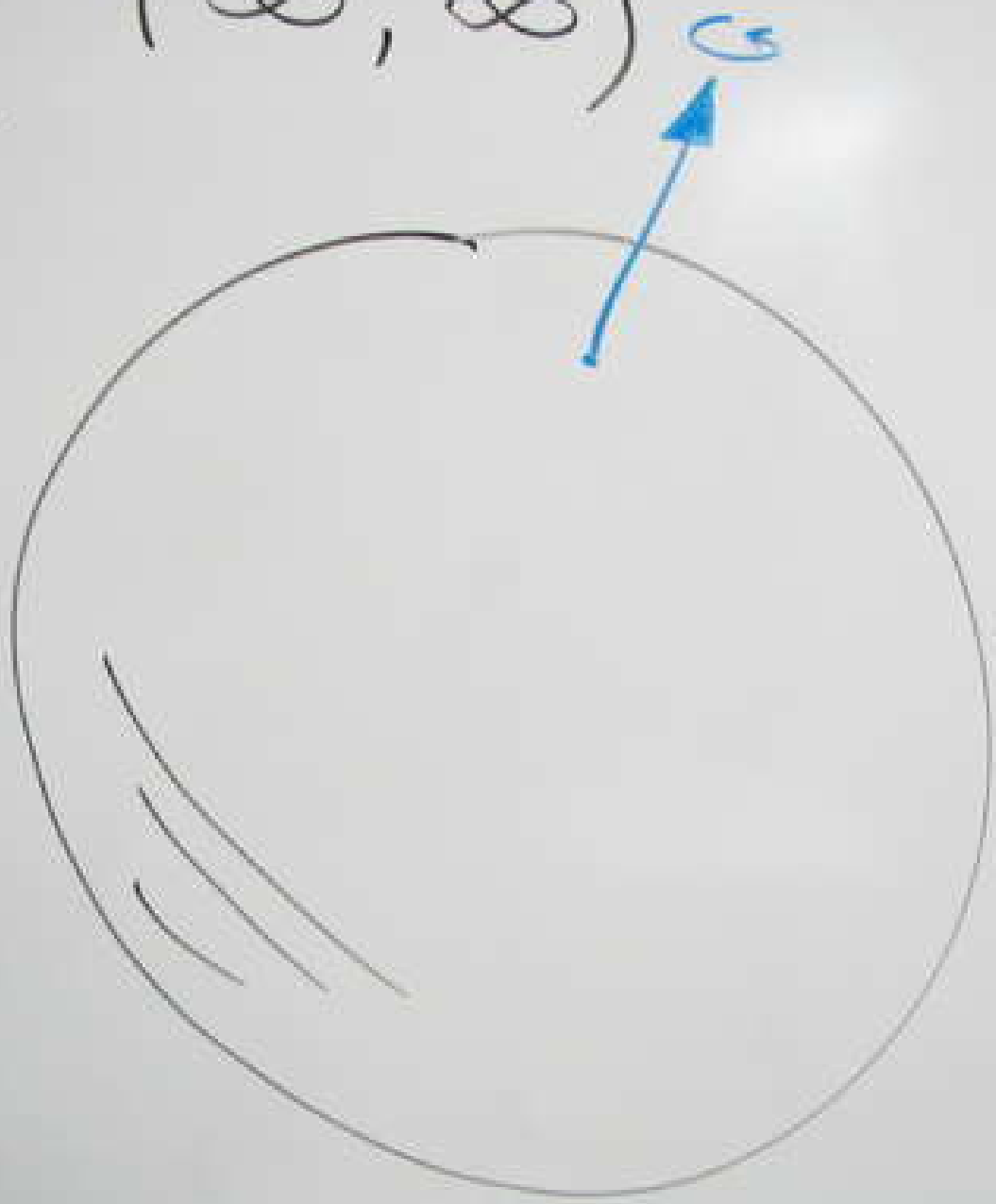


CASE I

(∞, ∞)

PROOF 1

(moving reference frame)

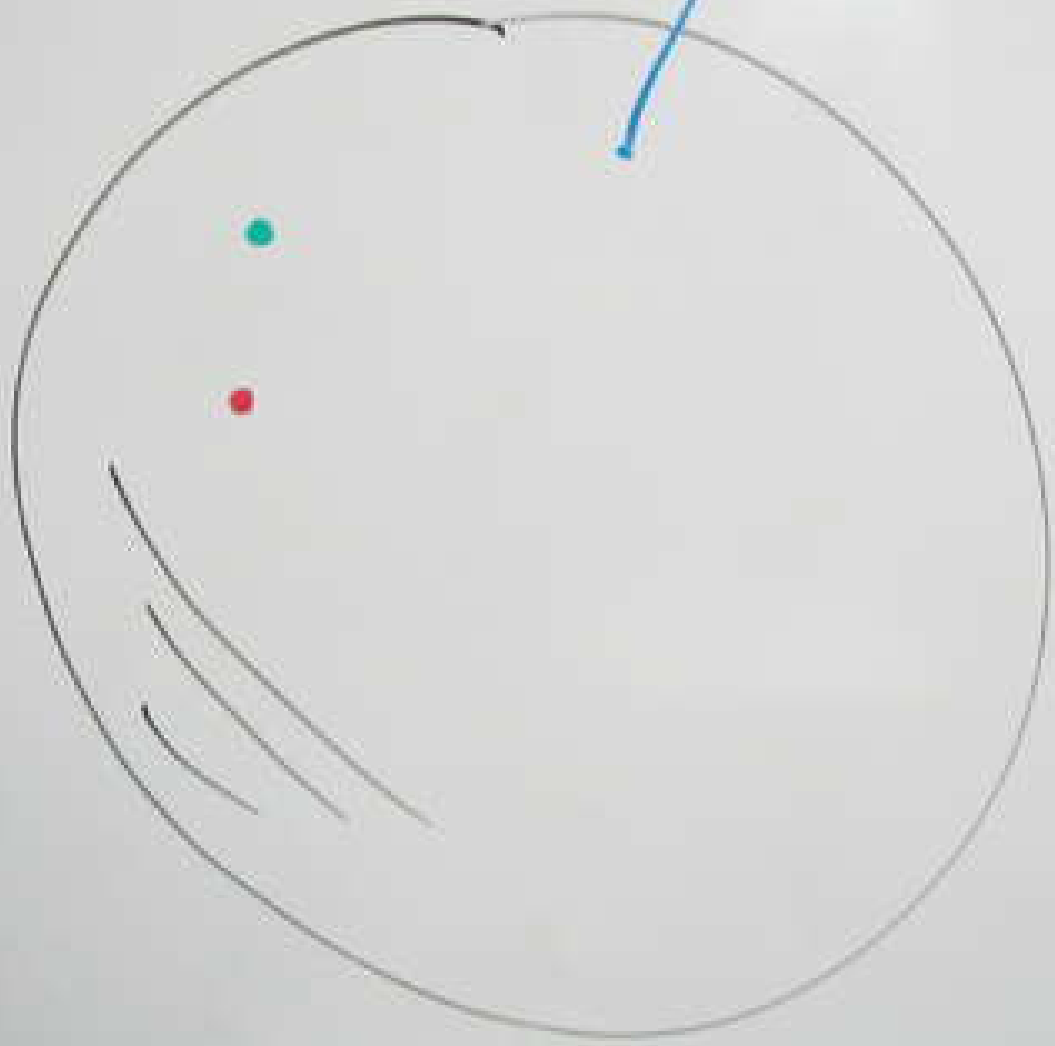


CASE I

(∞, ∞)

PROOF 1

(moving reference frame)

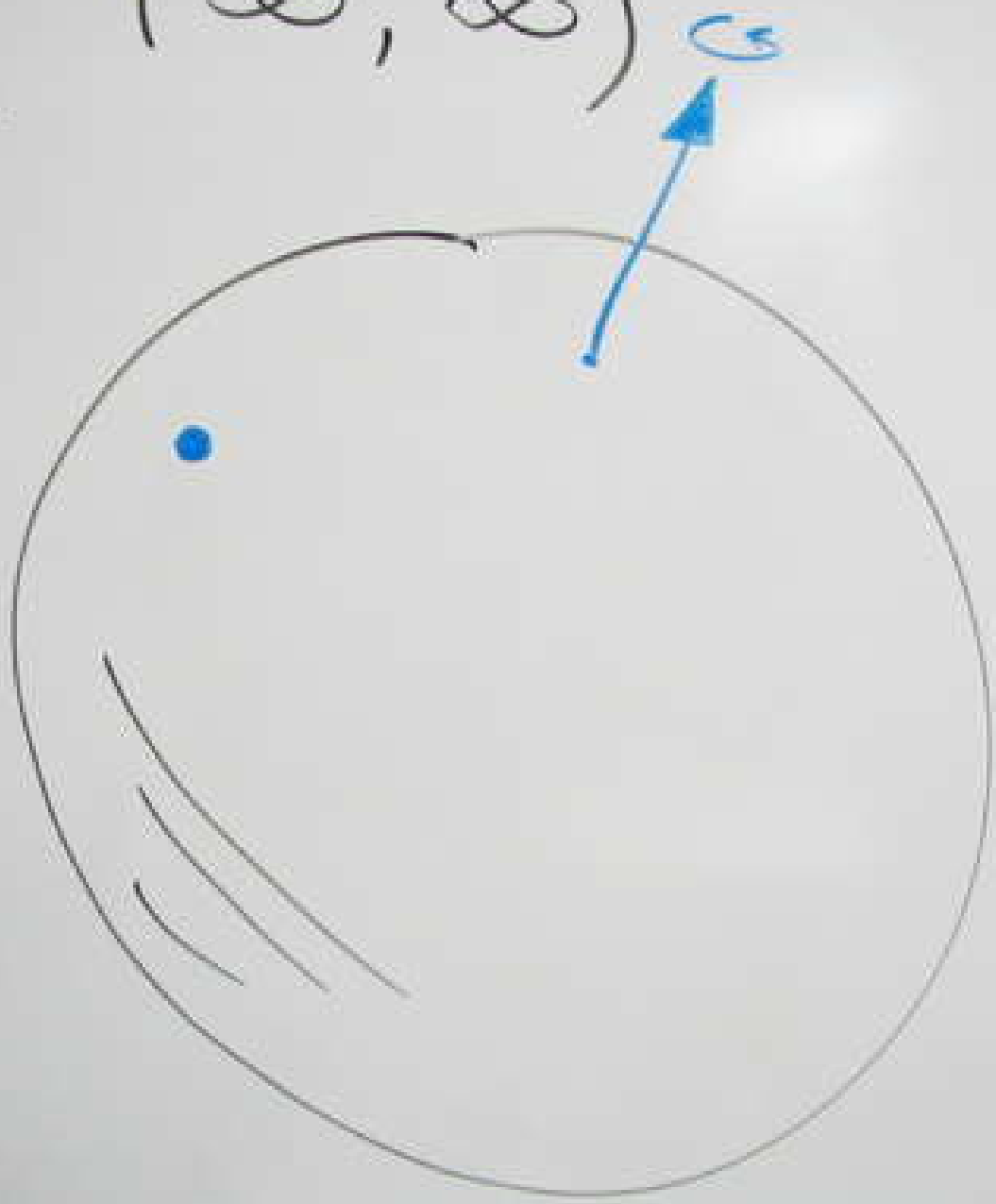


CASE I

(∞, ∞)

PROOF 1

(moving reference frame)

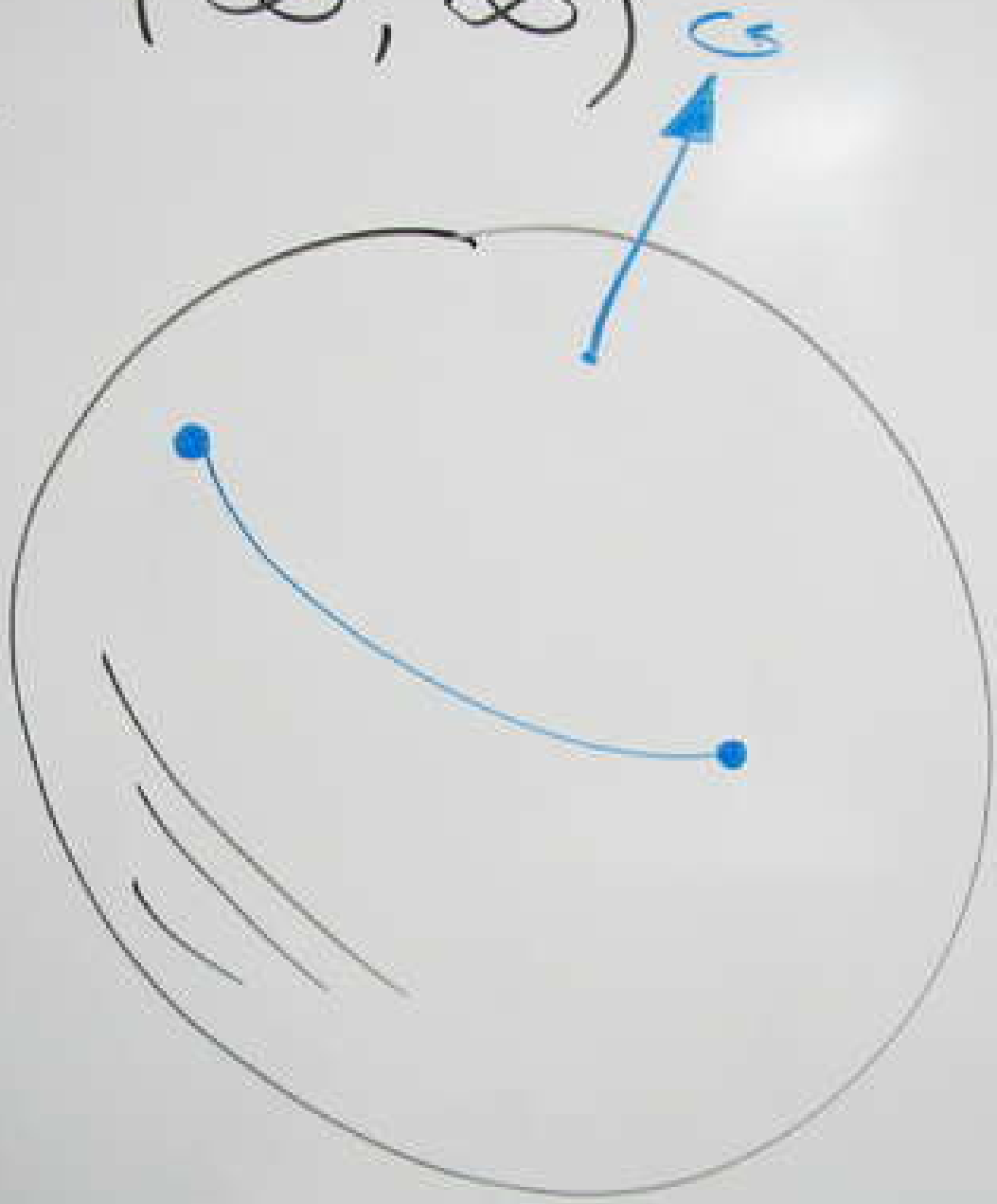


CASE I

(∞, ∞)

PROOF 1

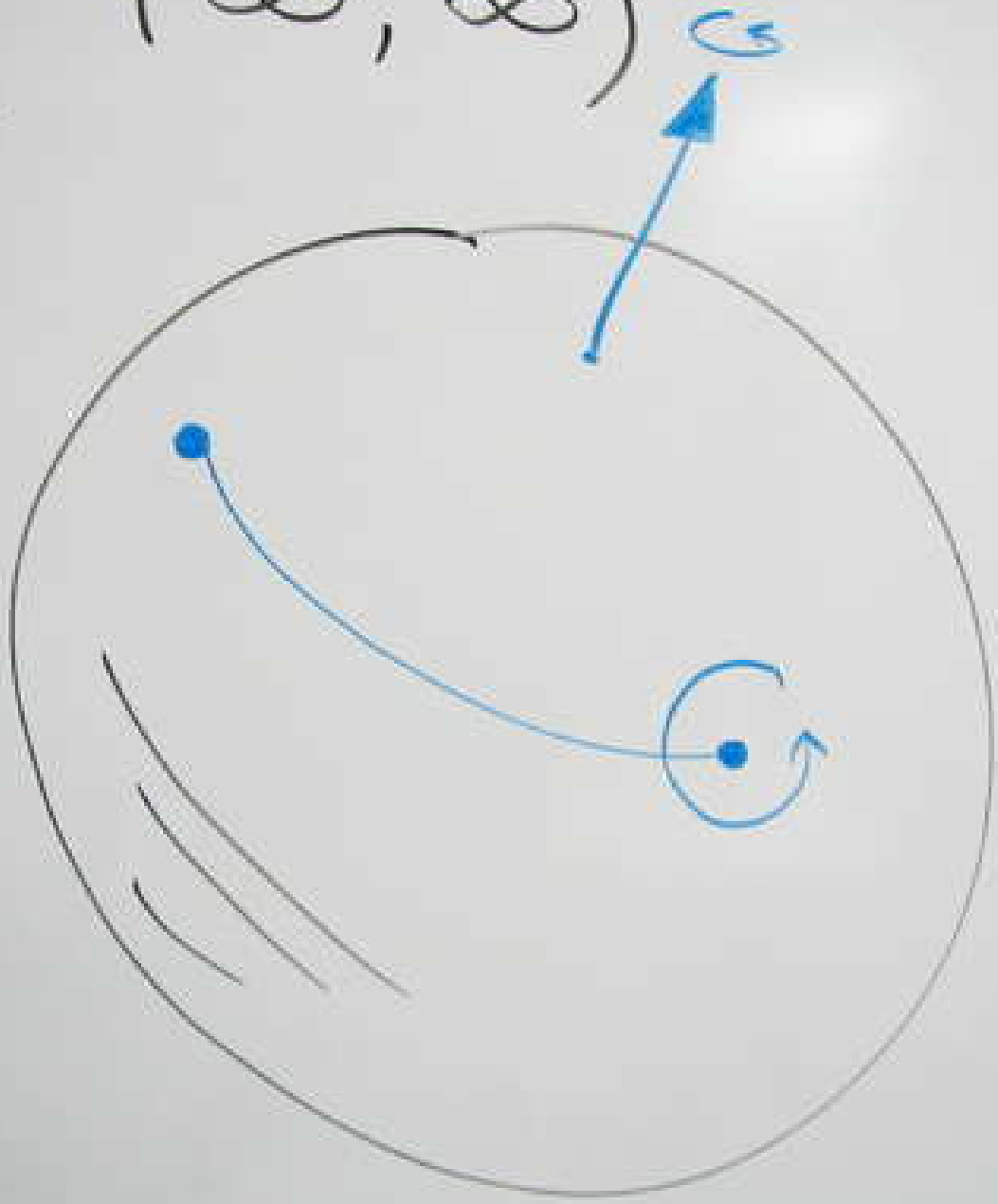
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

(moving reference frame)

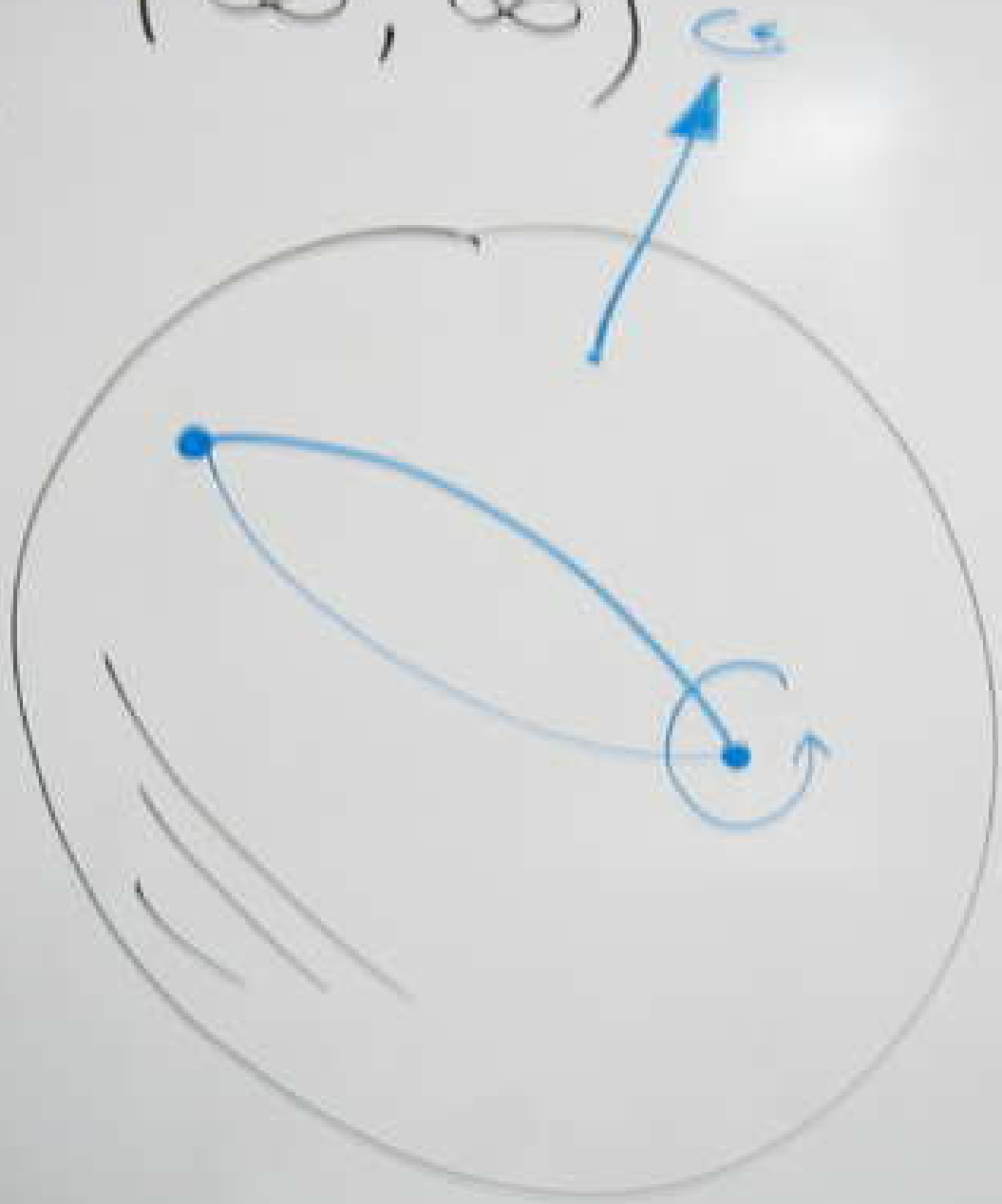


CASE I

(∞, ∞)

PROOF 1

(moving reference frame)

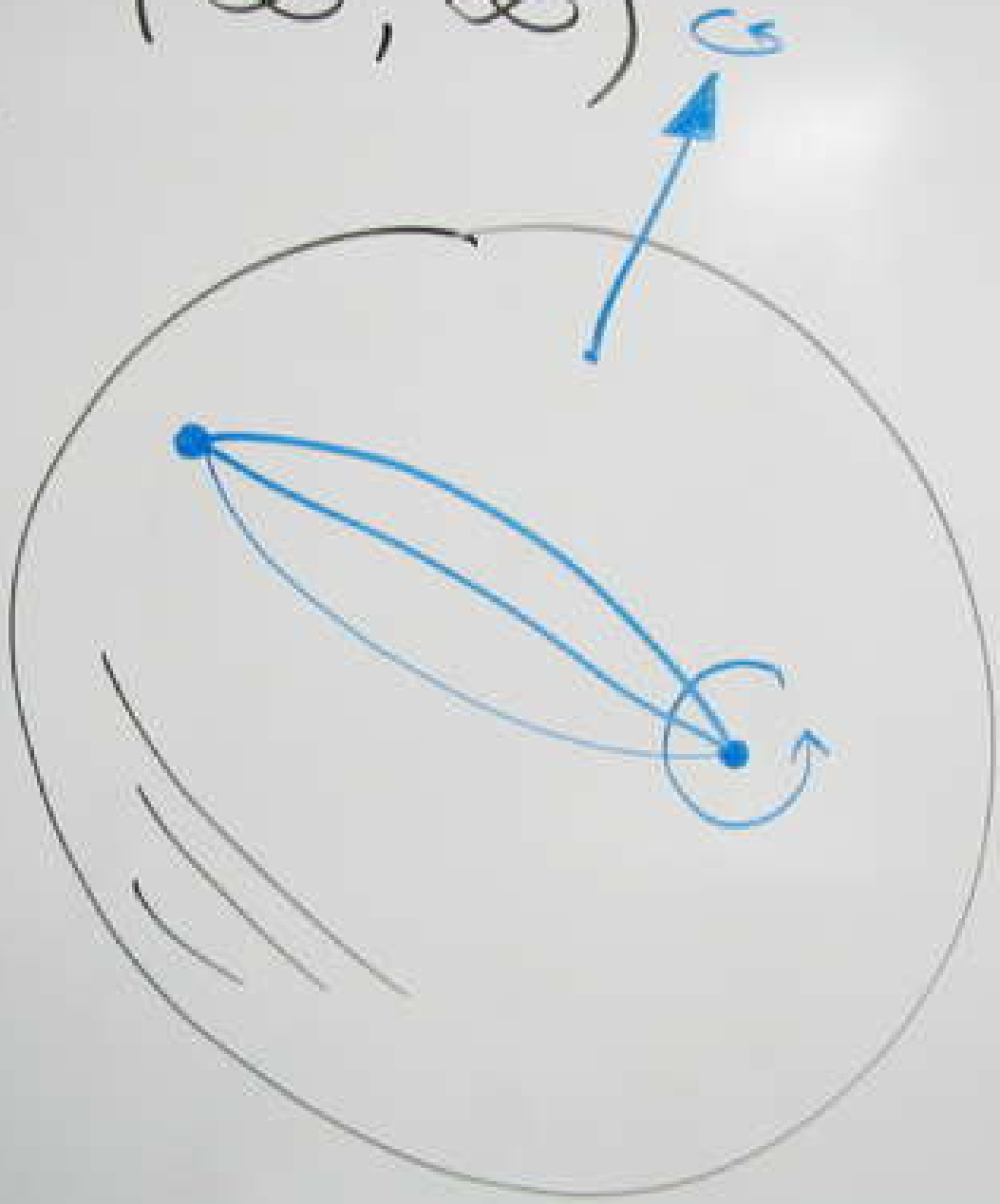


CASE I

(∞, ∞)

PROOF 1

(moving reference frame)

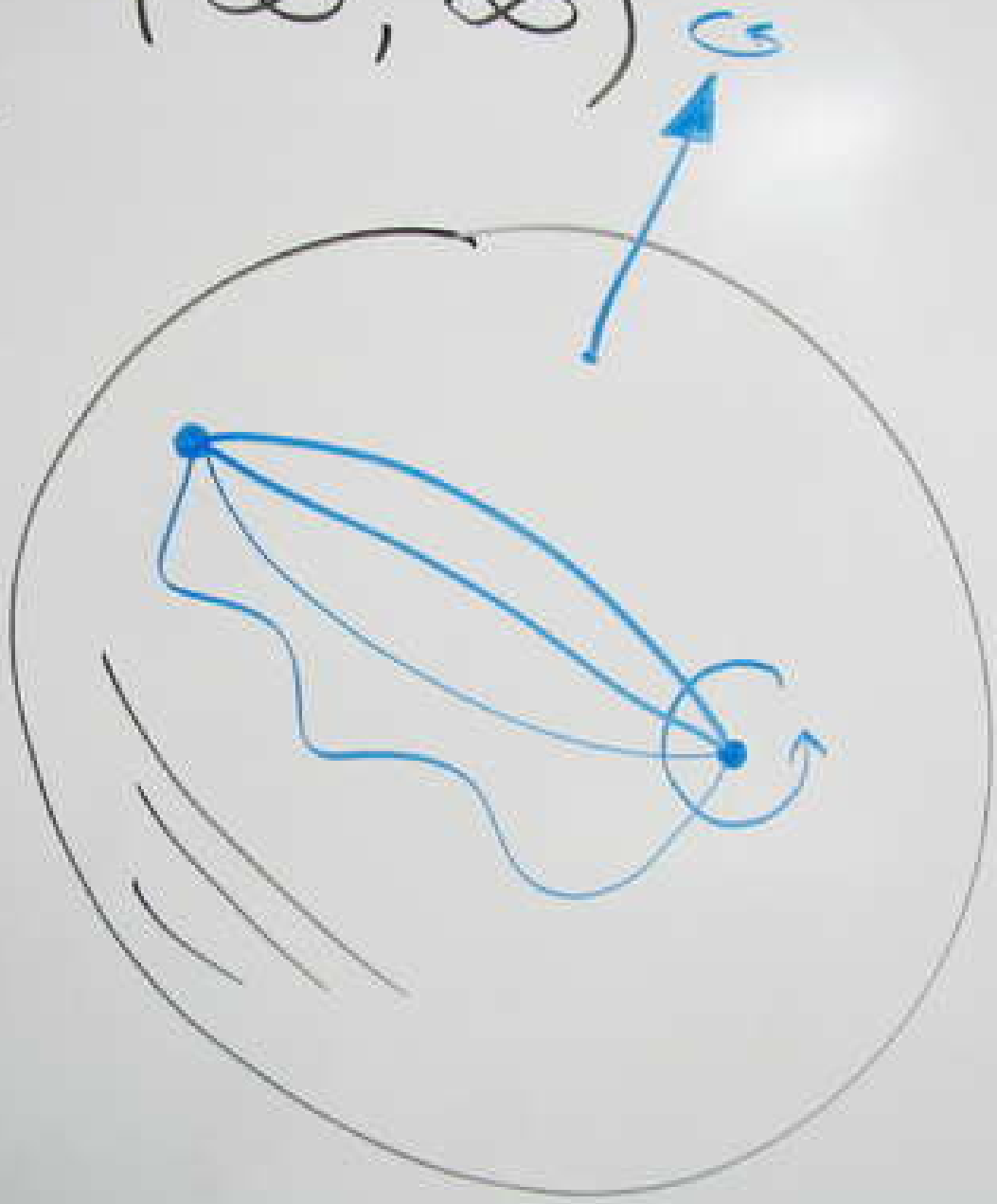


CASE I

(∞, ∞)

PROOF 1

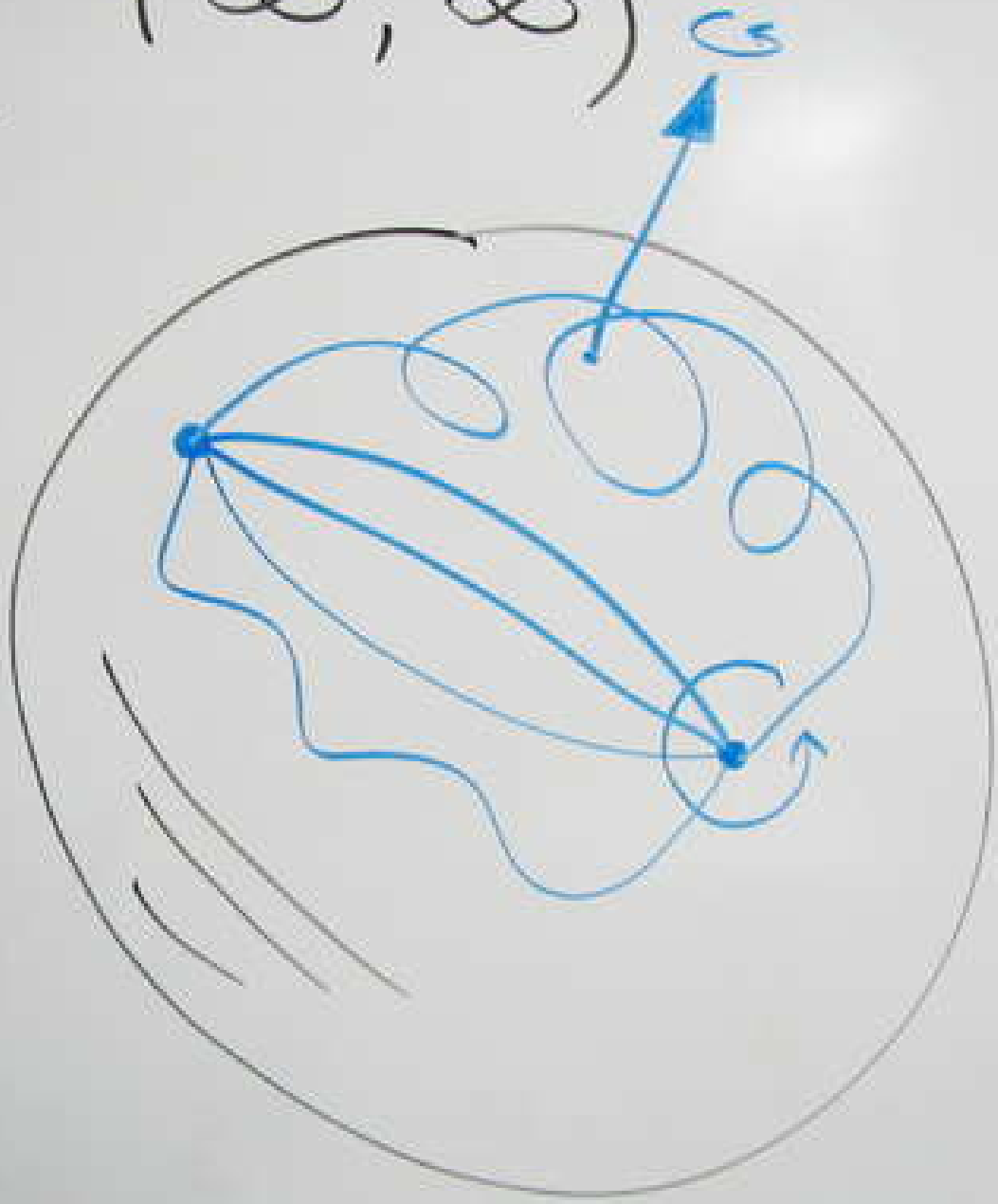
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

(moving reference frame)

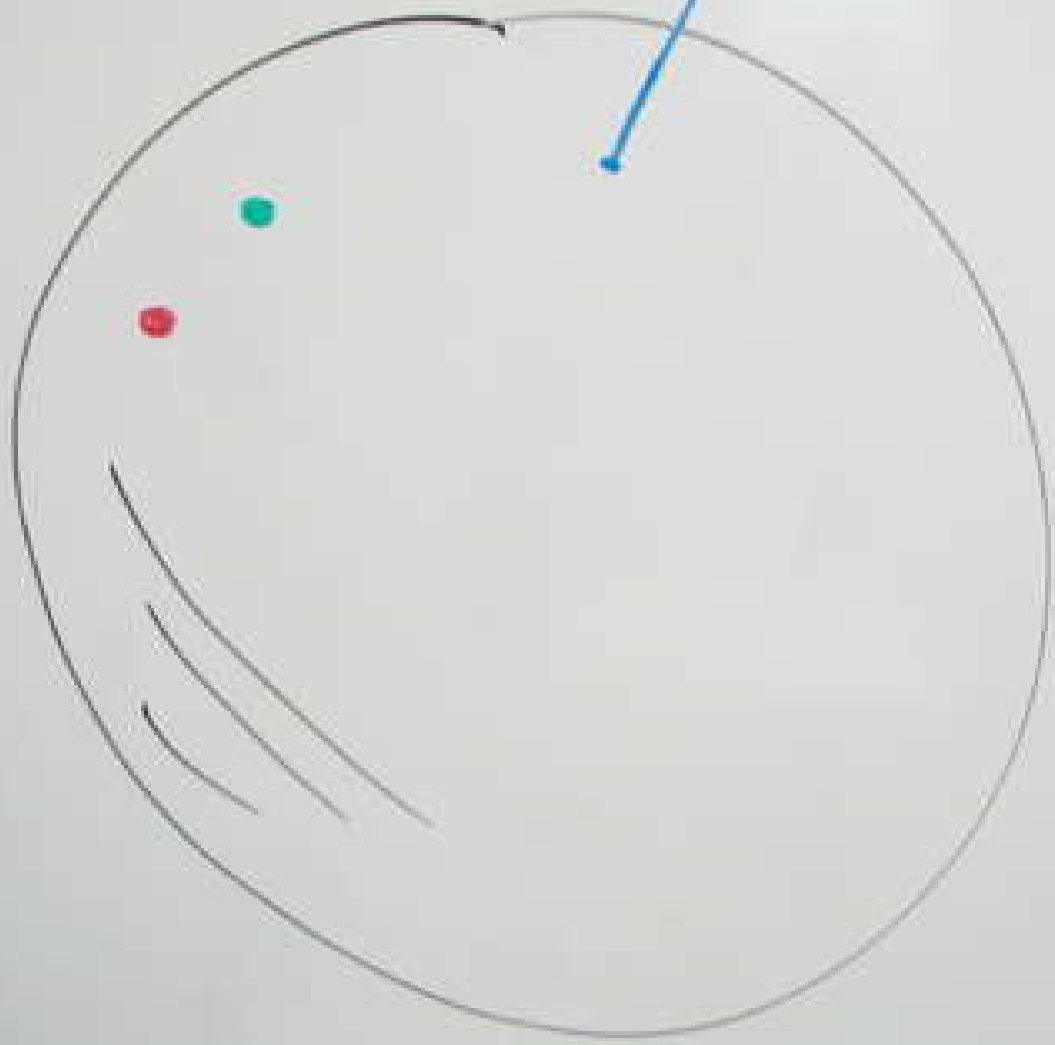


CASE I

(∞, ∞)

PROOF 1

(moving reference frame)

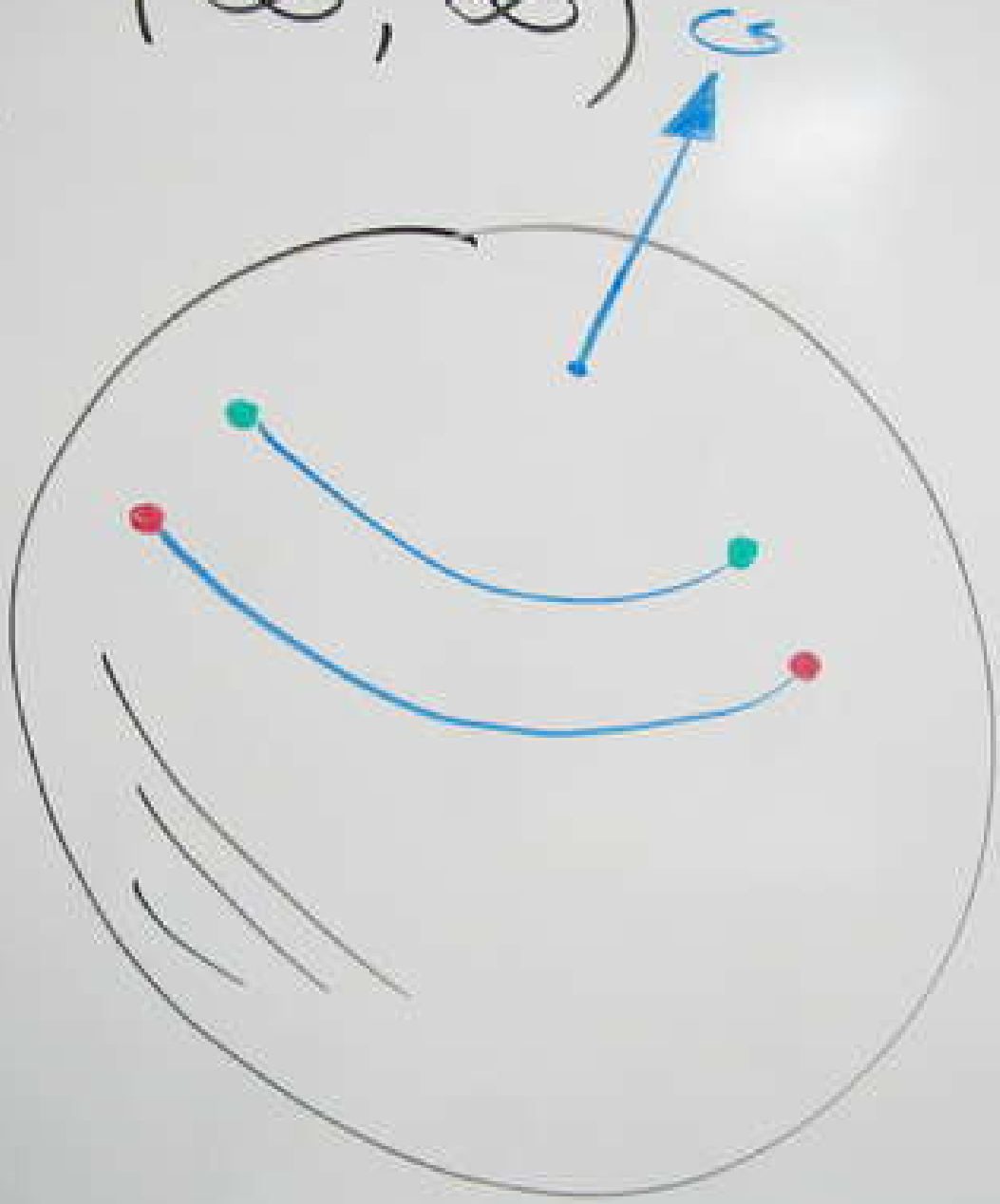


CASE I

(∞, ∞)

PROOF 1

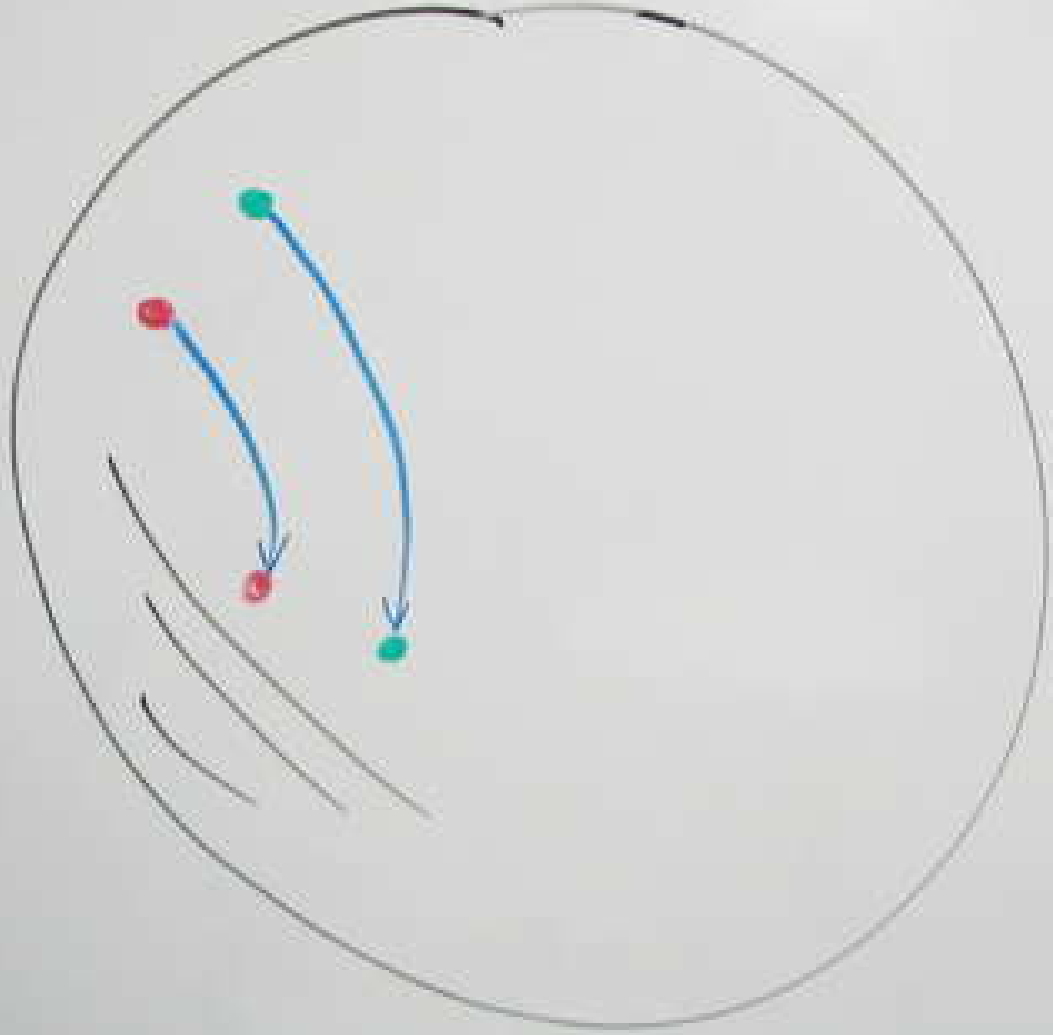
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

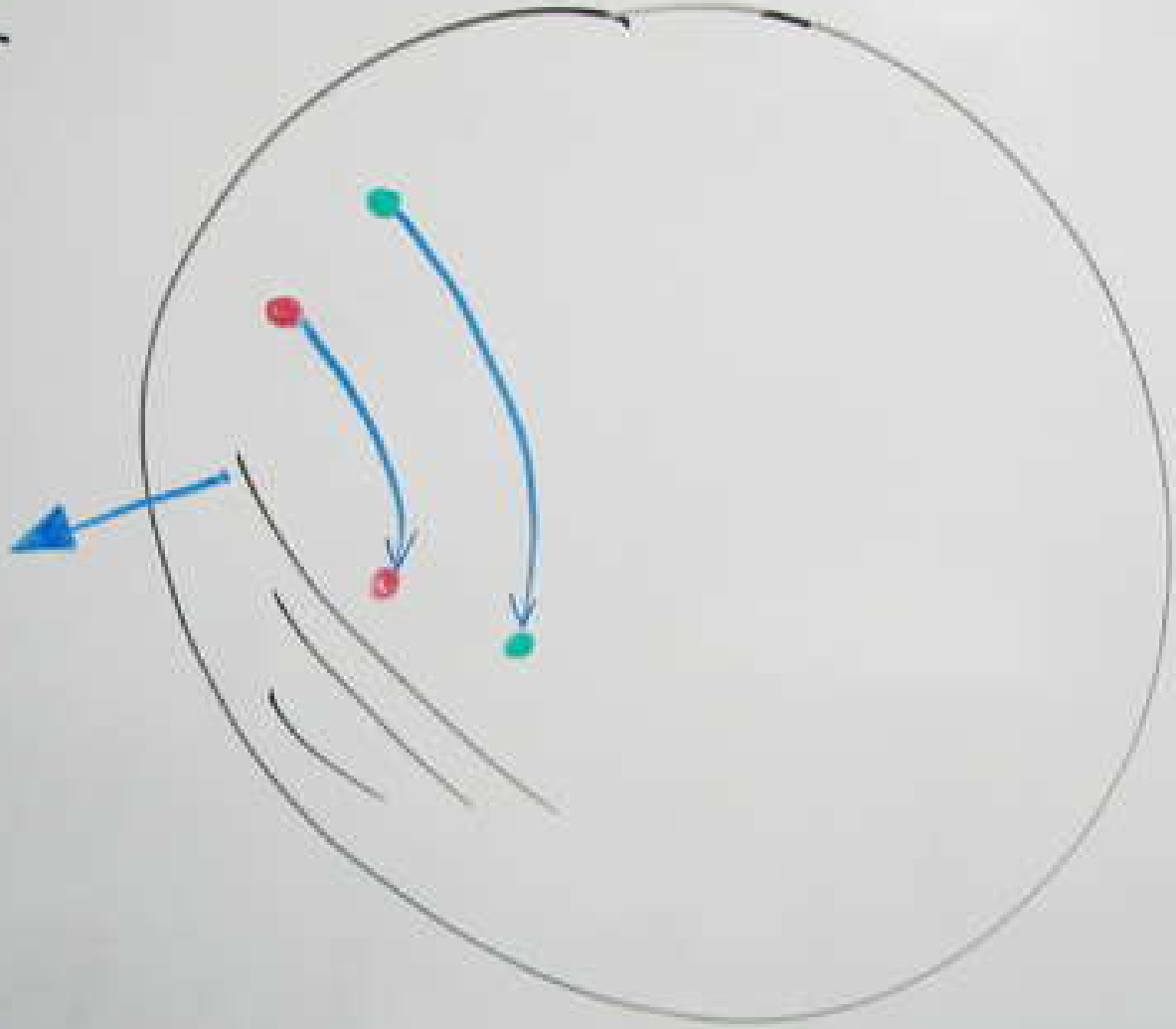
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

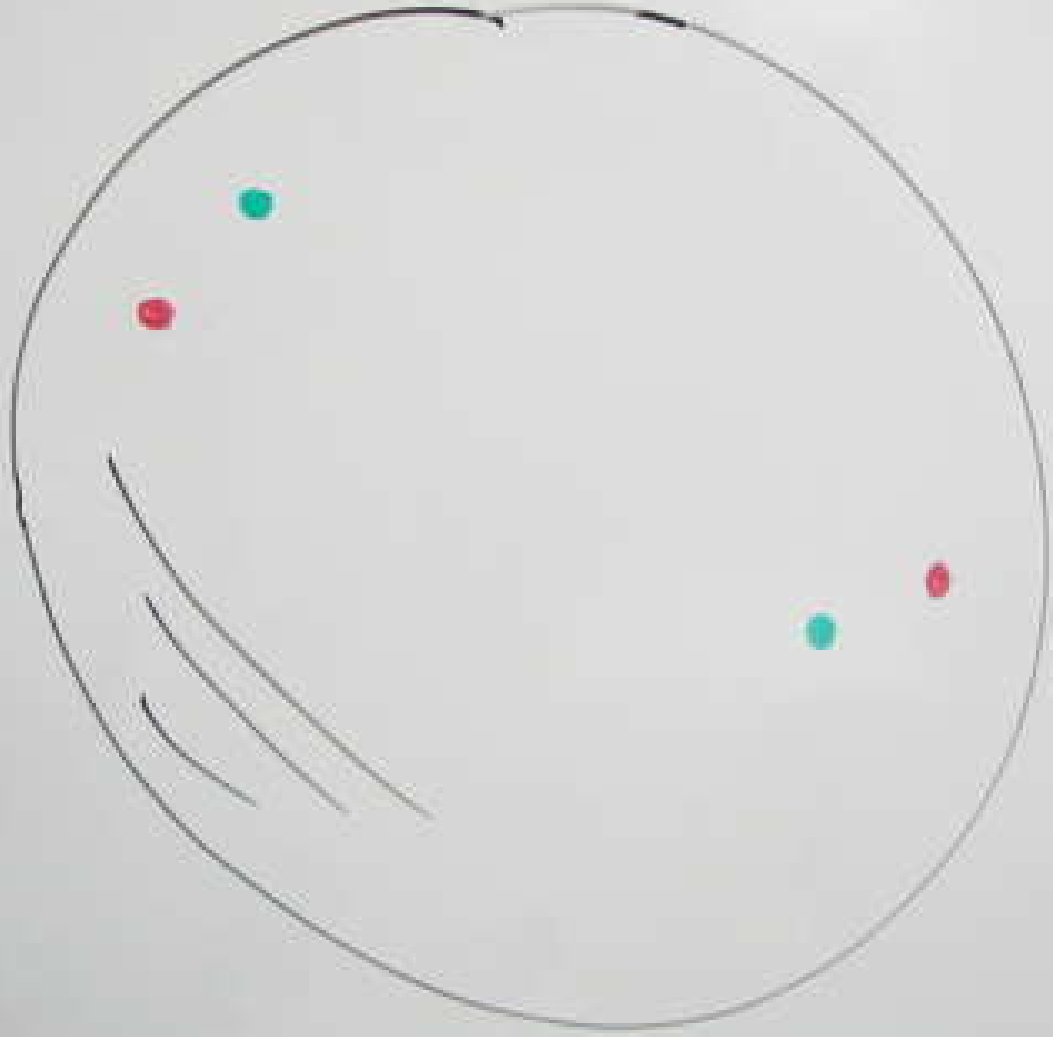
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

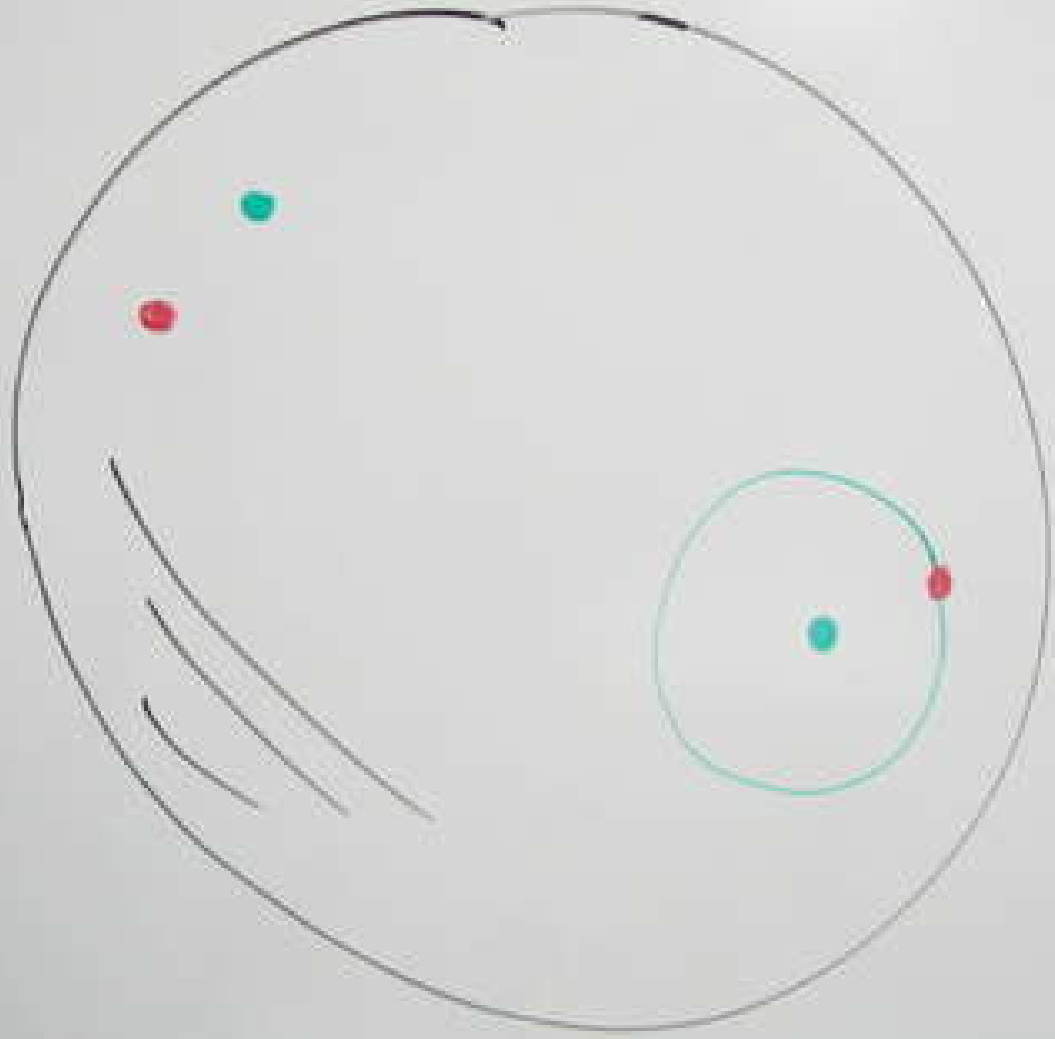
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

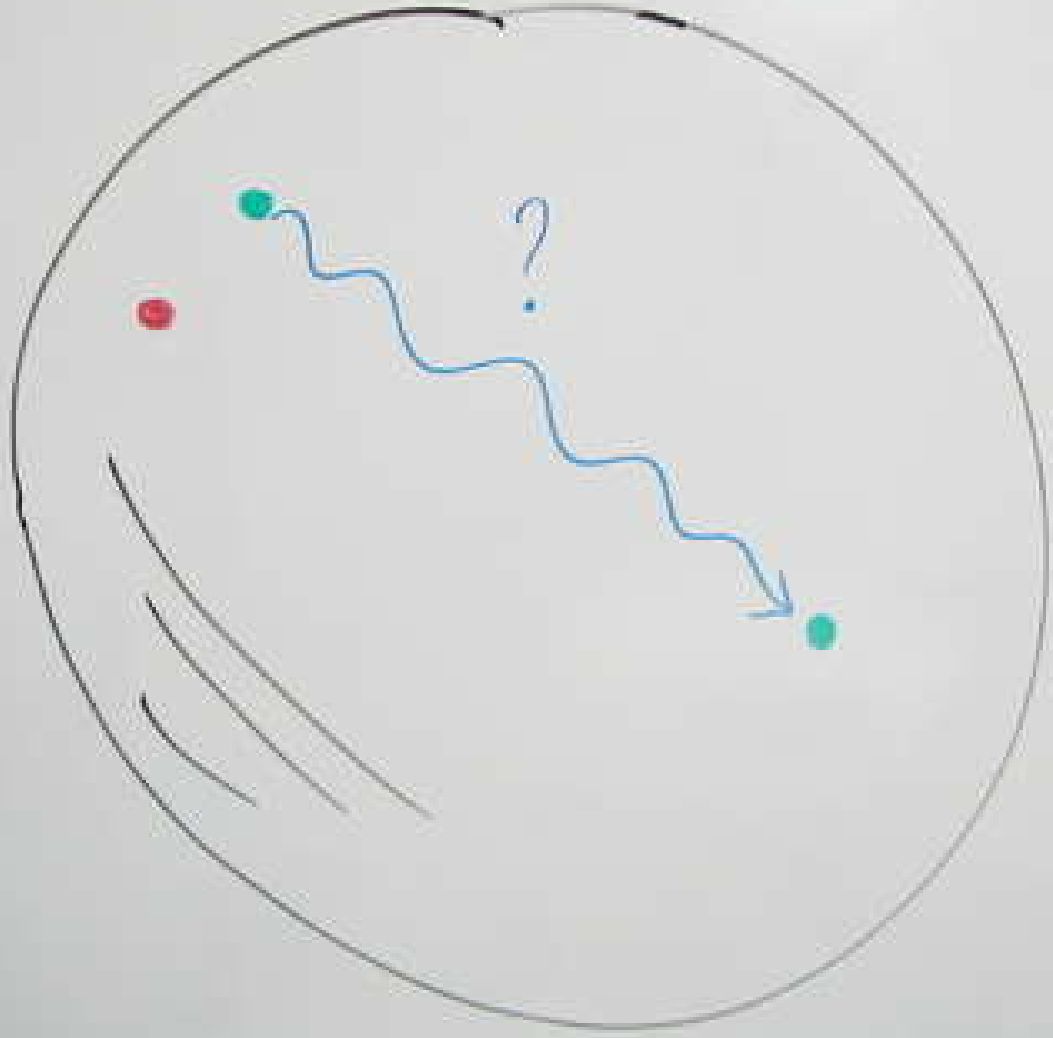
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

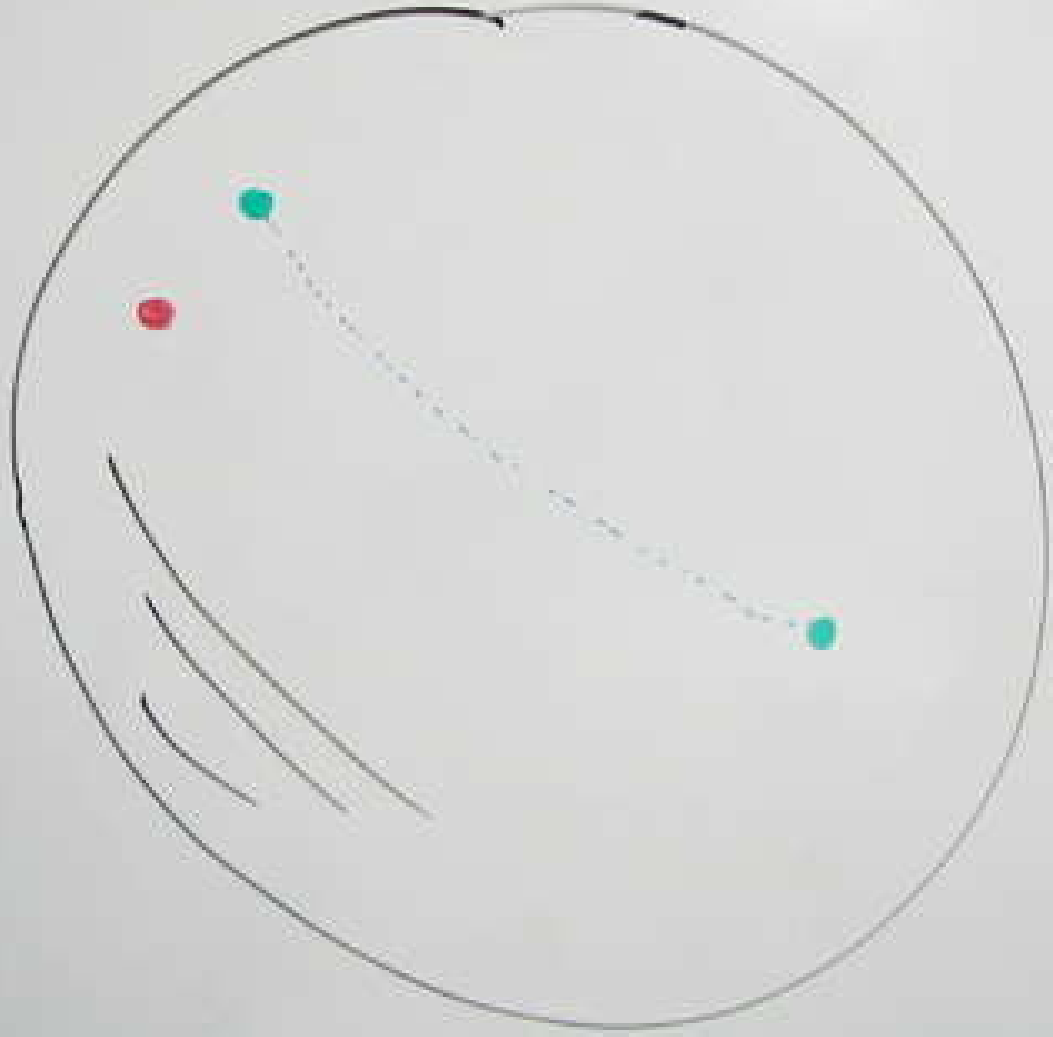
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

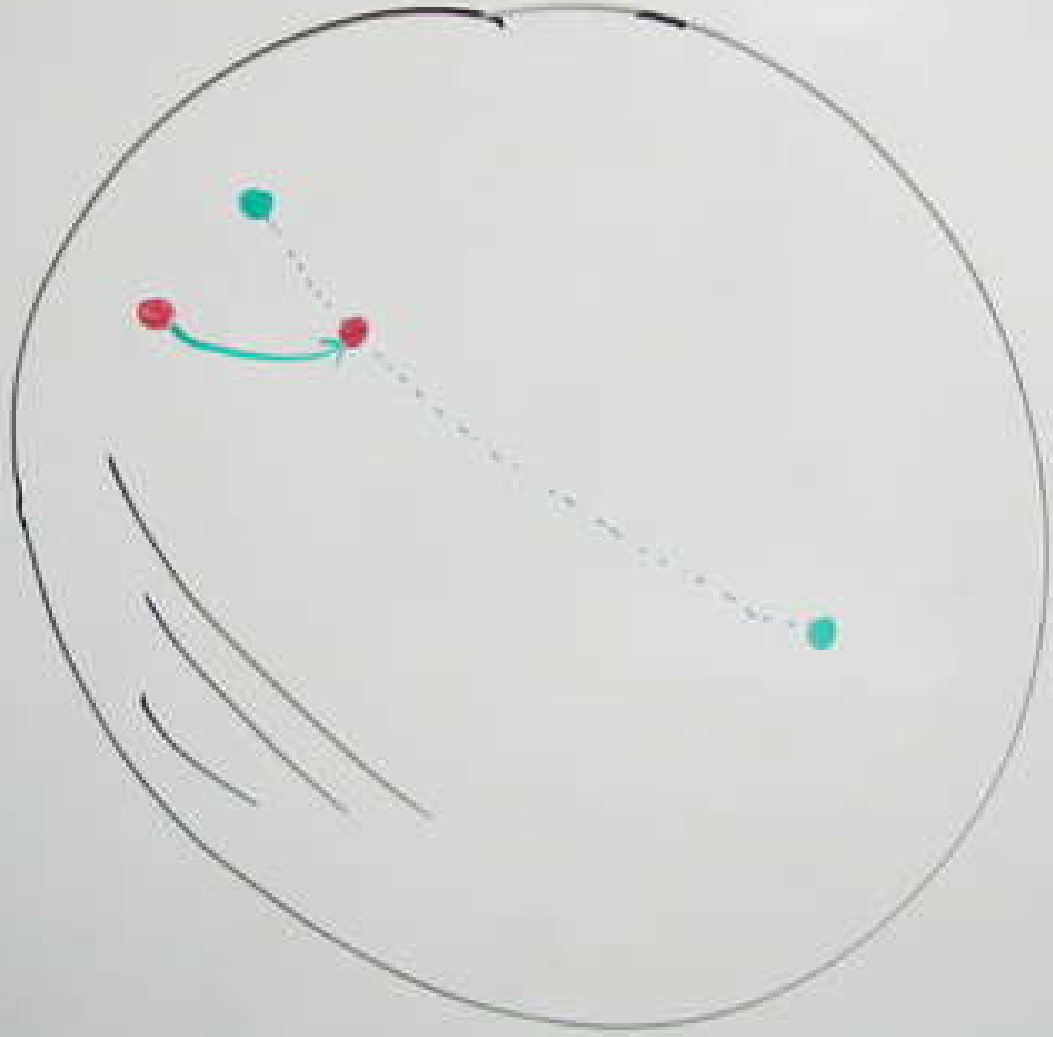
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

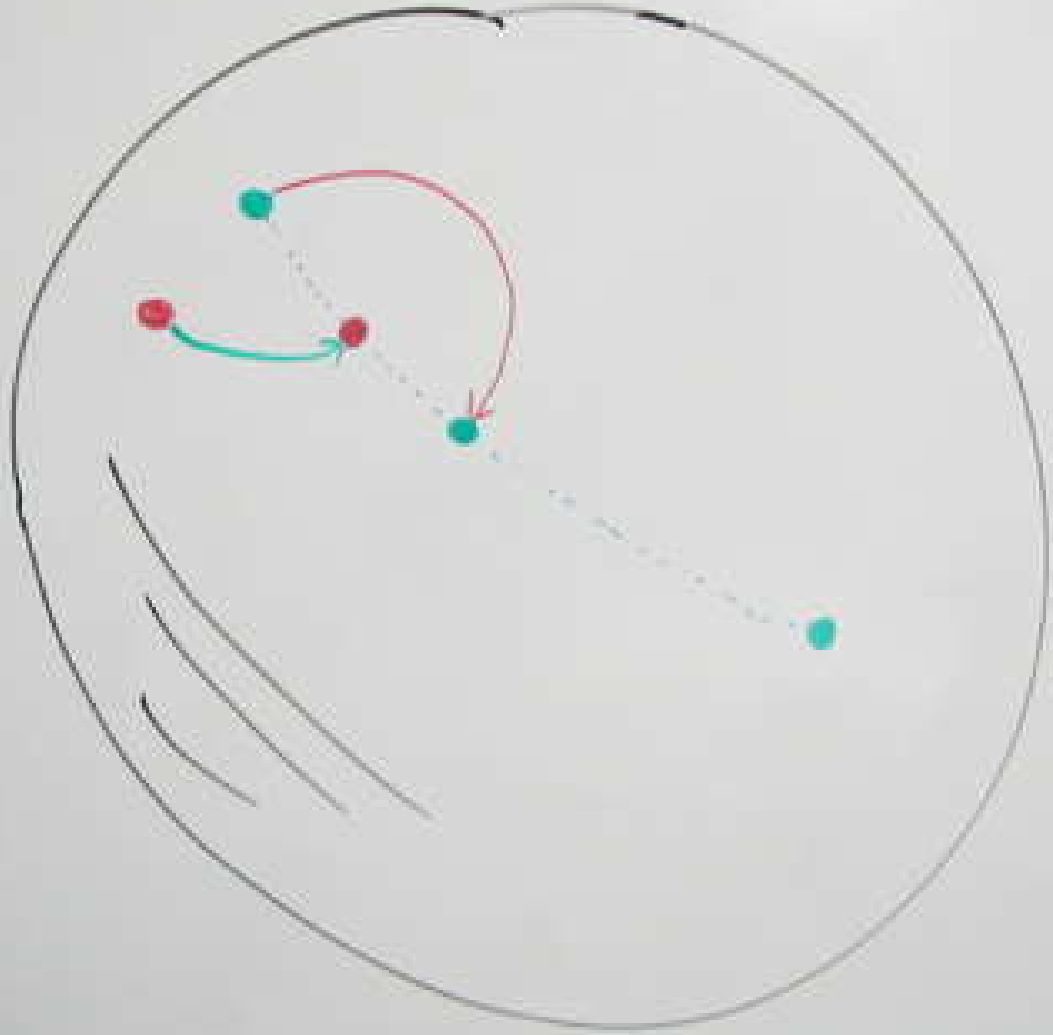
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

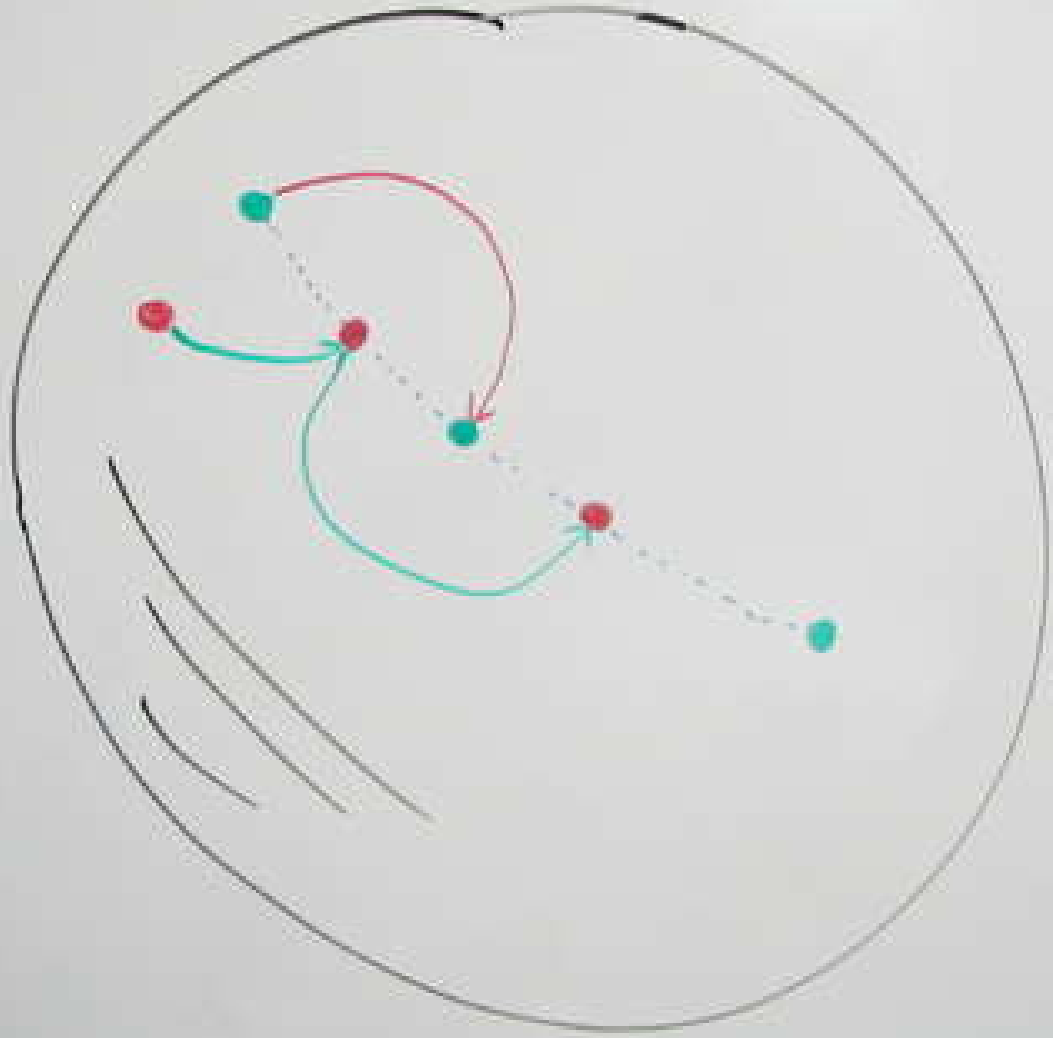
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

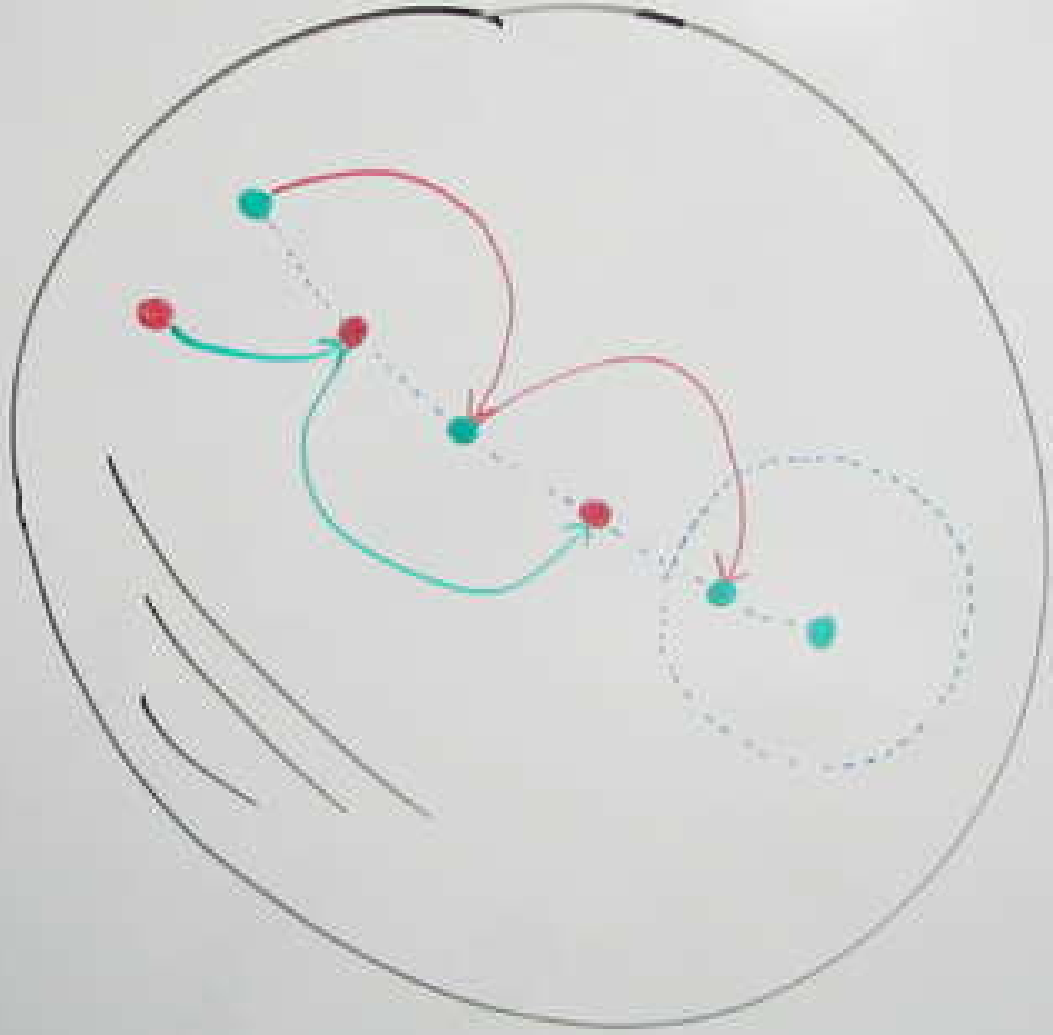
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

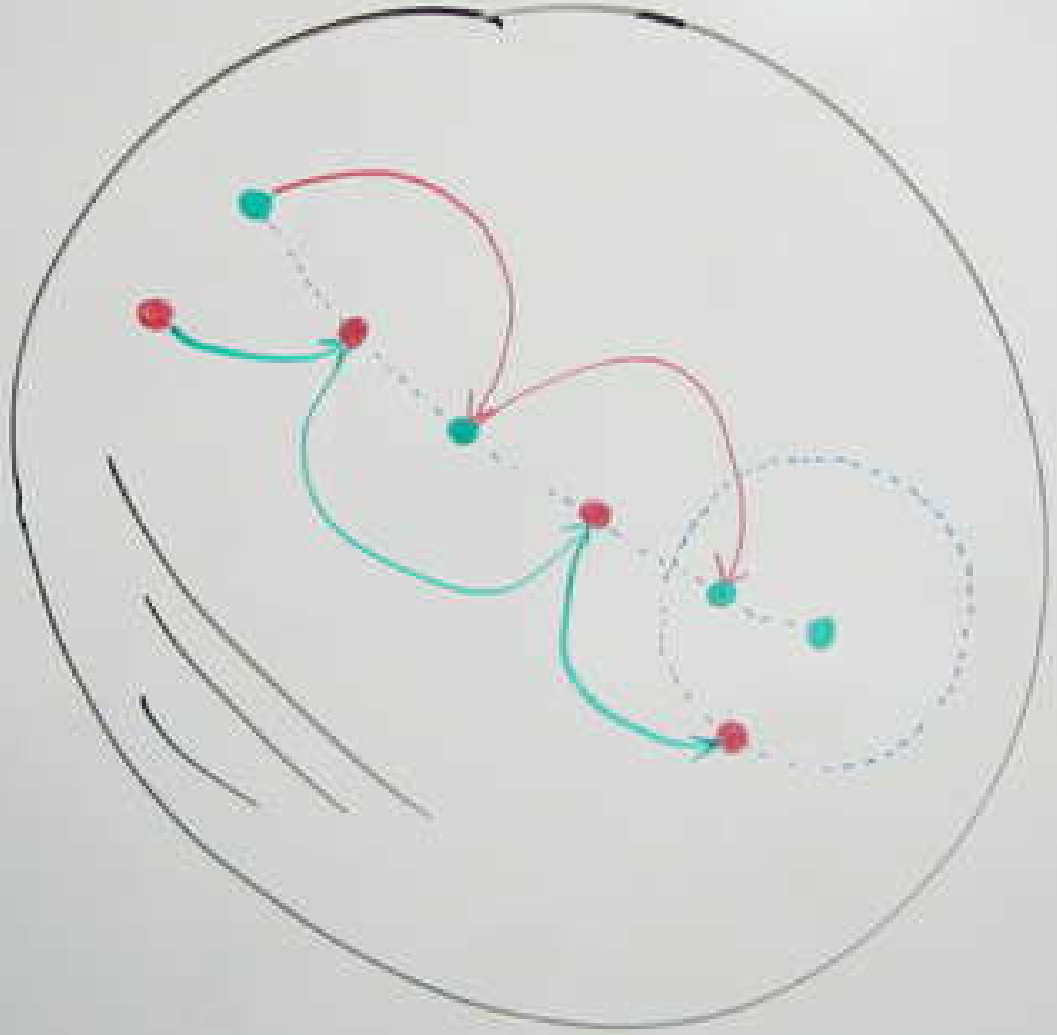
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

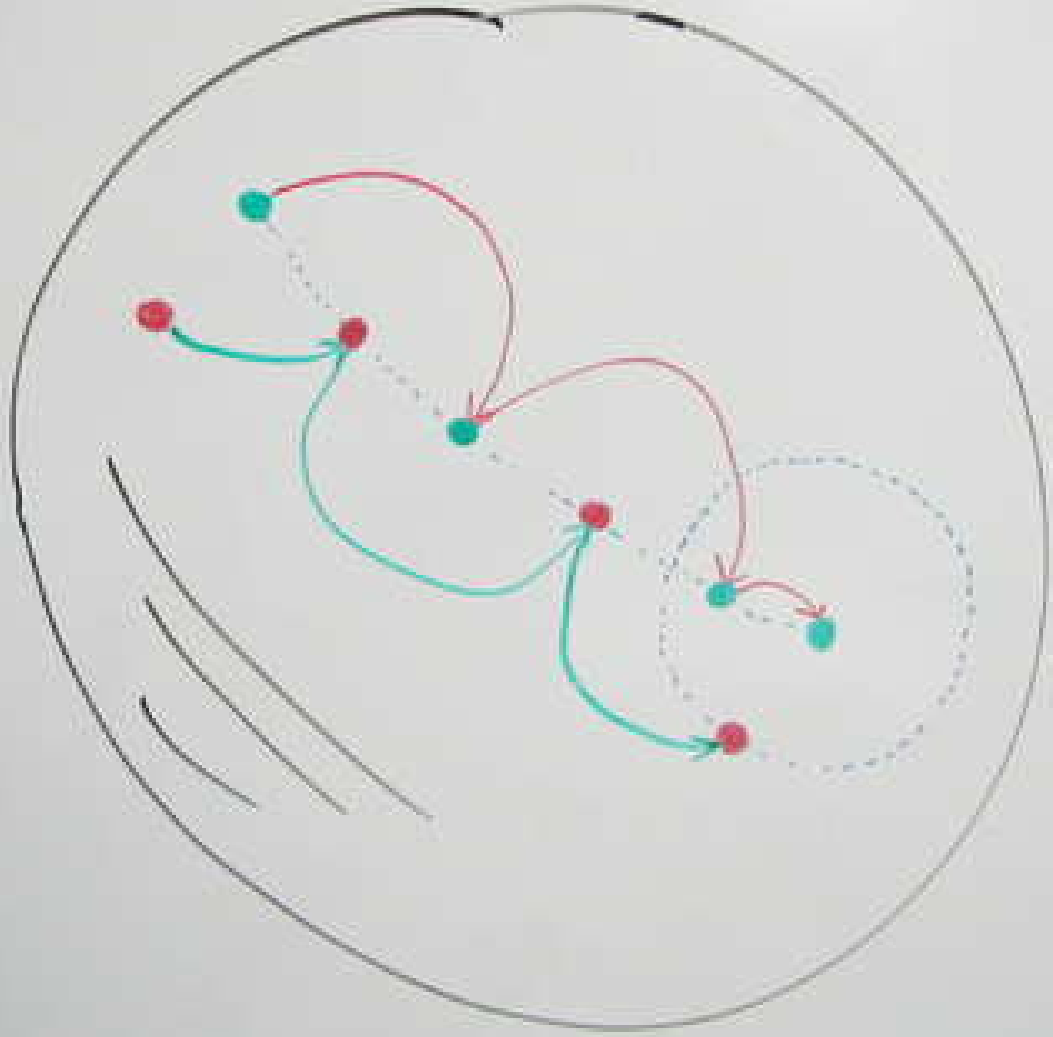
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

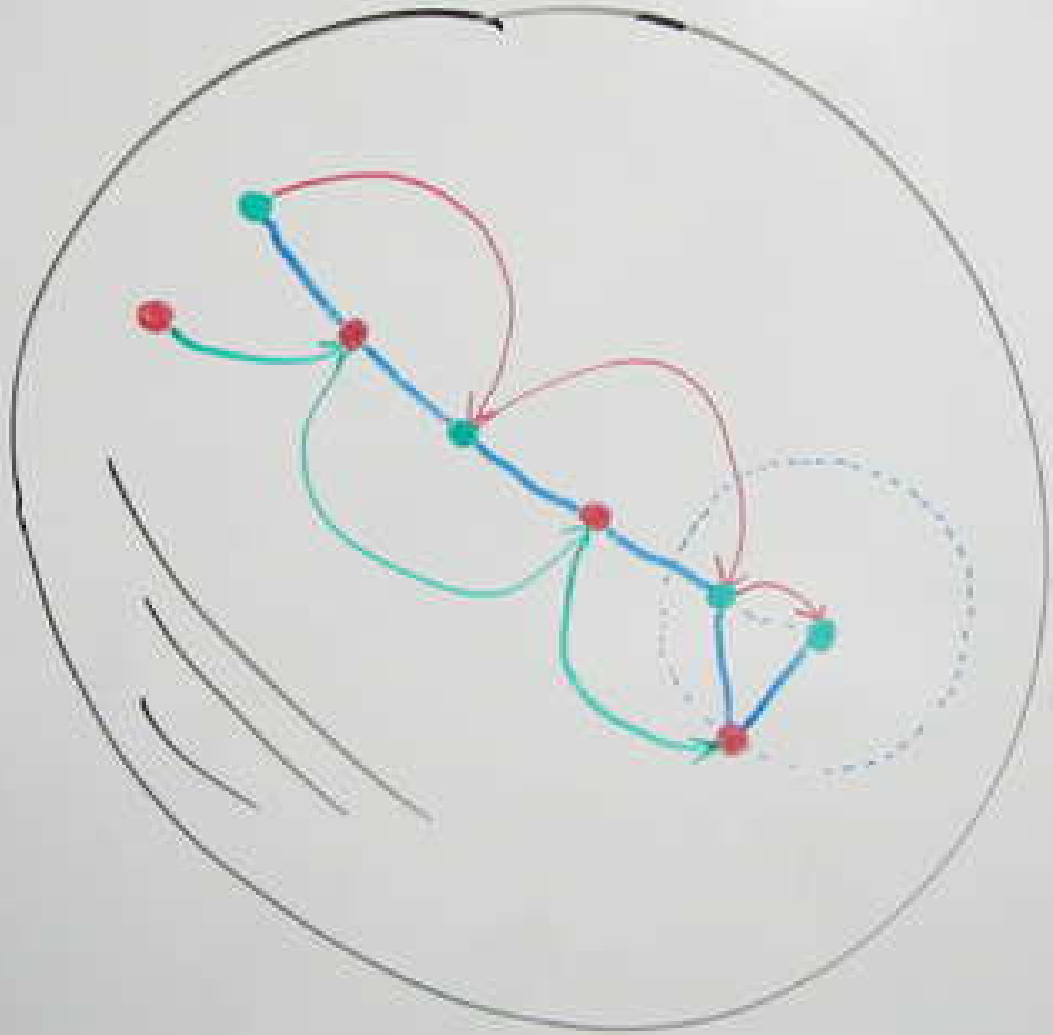
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

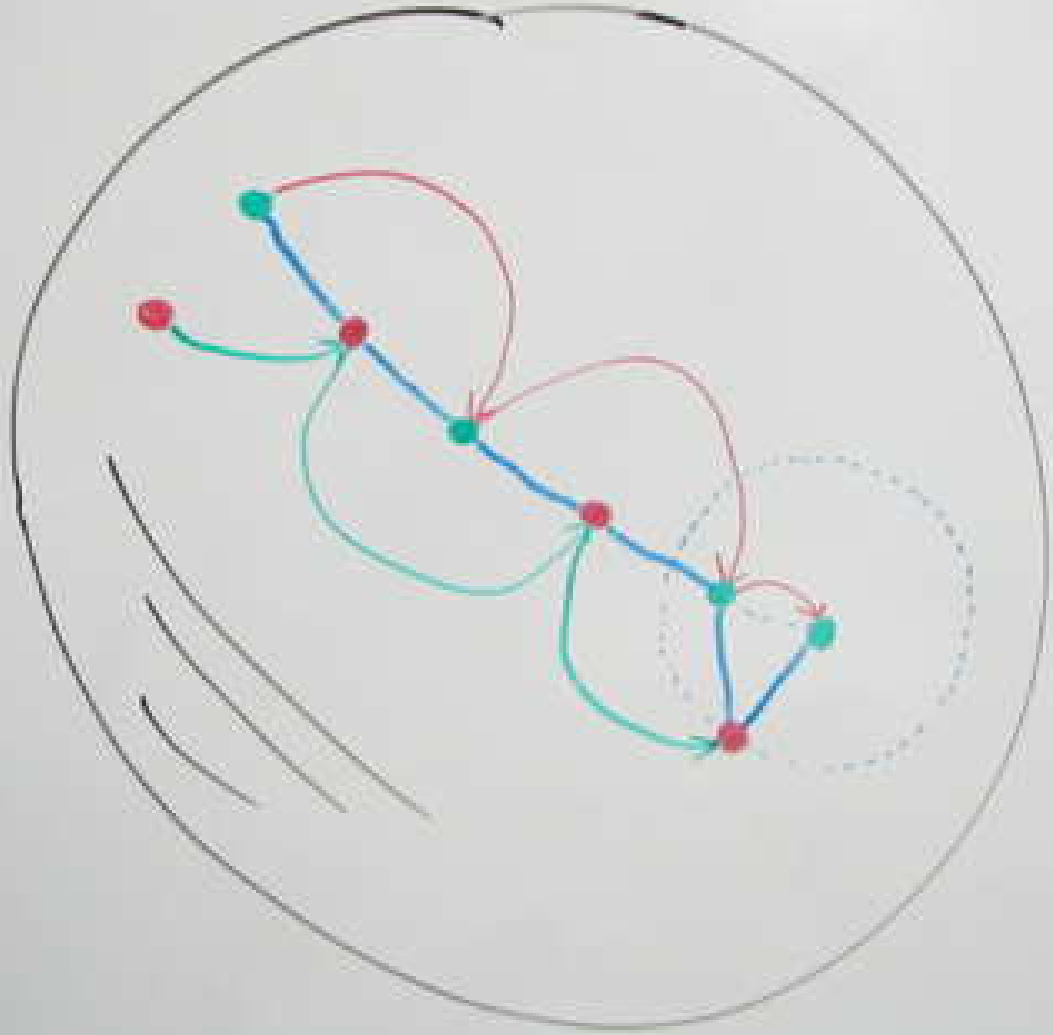
(moving reference frame)



CASE I (∞, ∞)

PROOF 1

(moving reference frame)

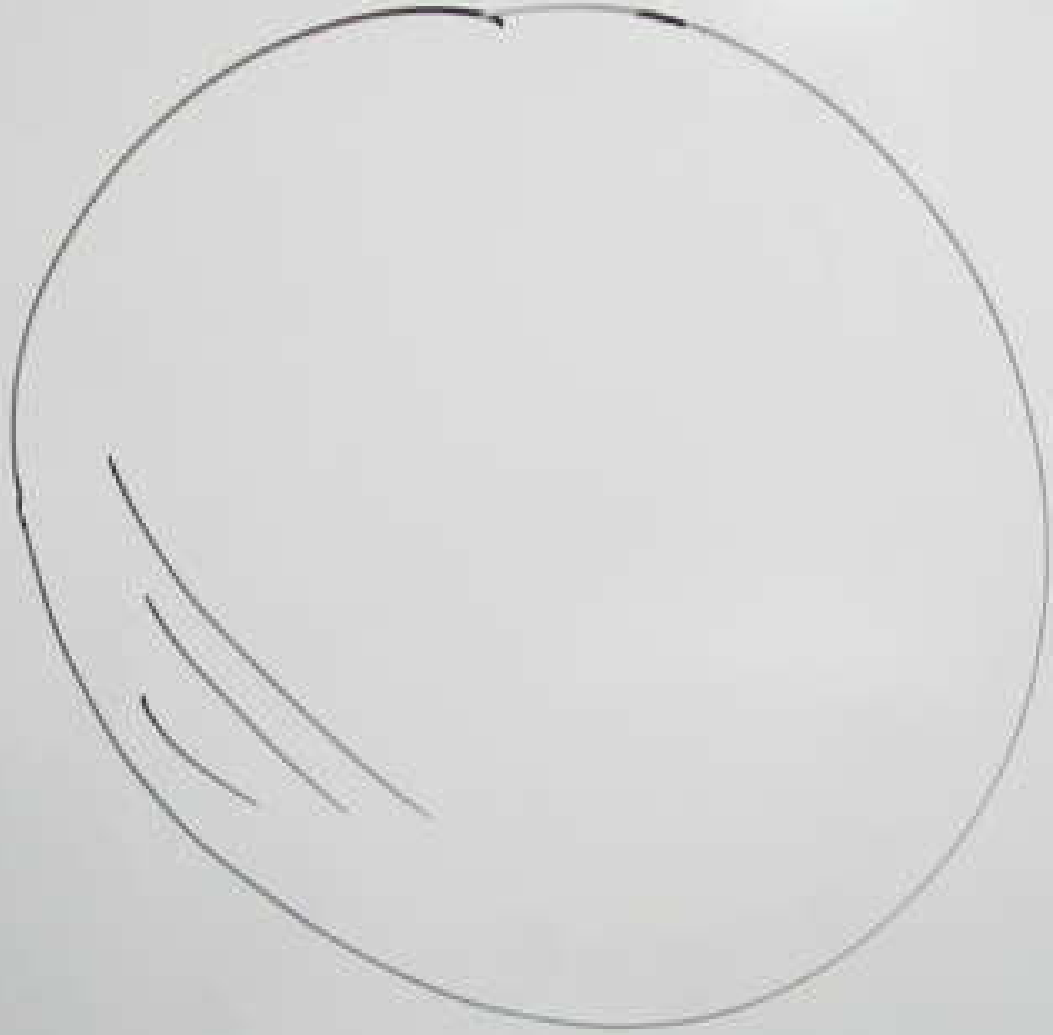


CASE I

(∞, ∞) $\theta = \frac{\pi}{2}$?

PROOF 1

(moving reference frame)

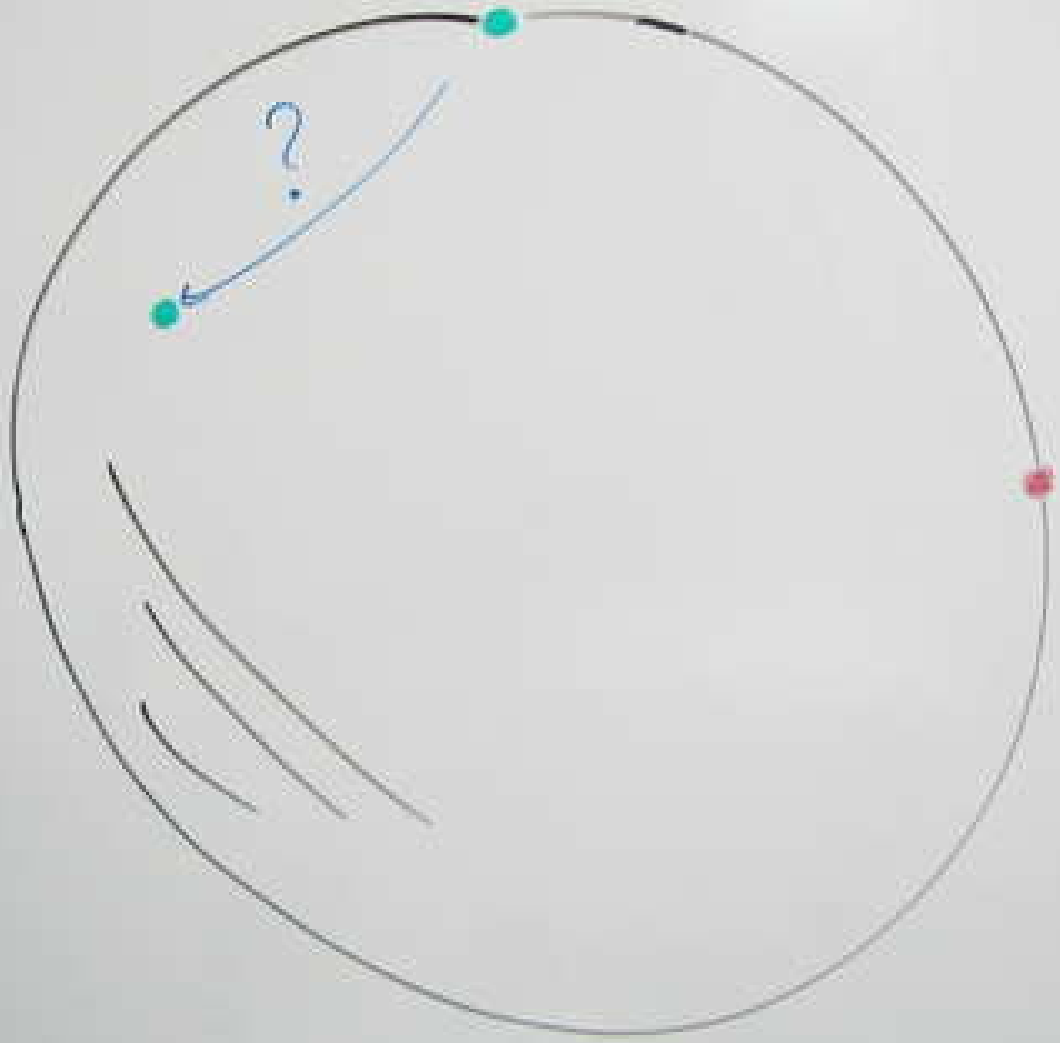


CASE I

(∞, ∞) $\theta = \frac{\pi}{2}$?

PROOF 1

(moving reference frame)

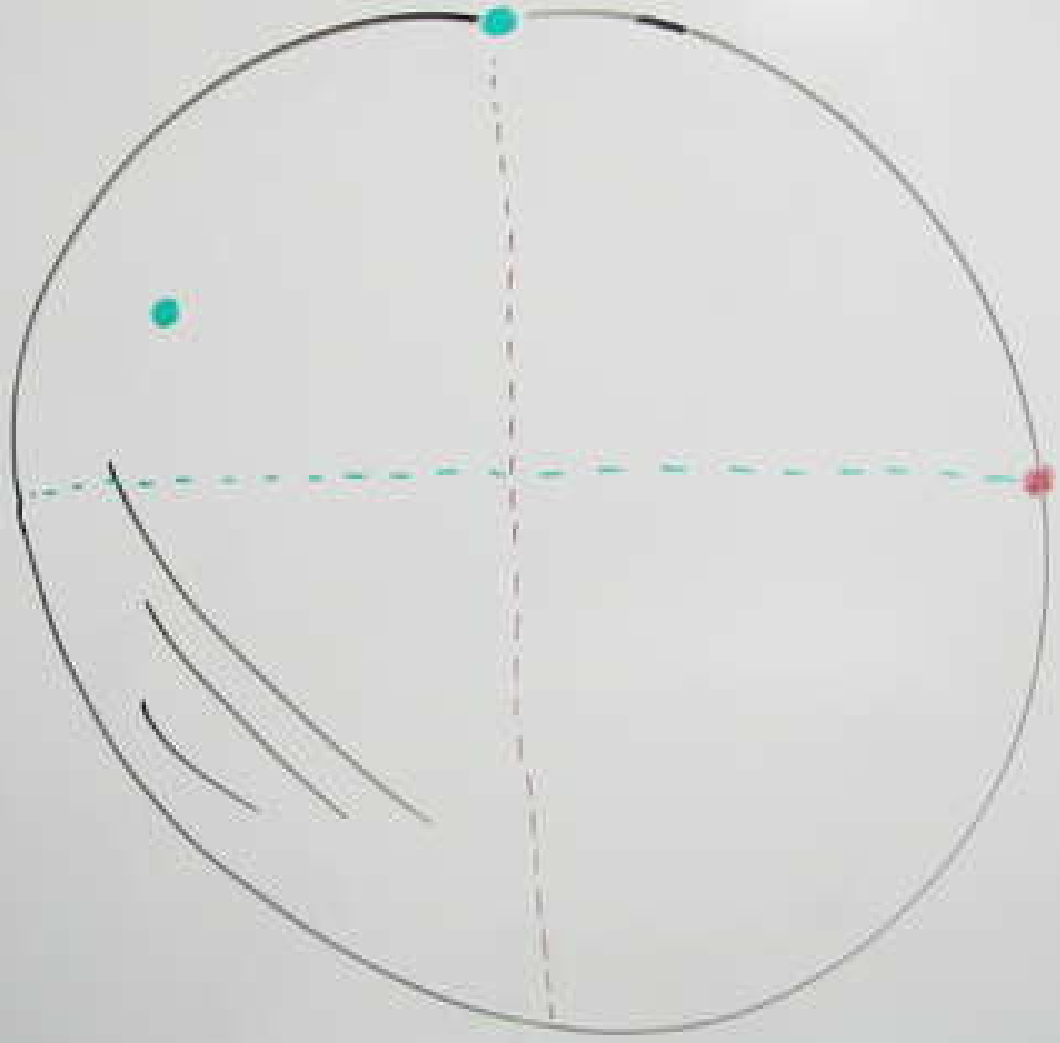


CASE I

(∞, ∞) $\theta = \frac{\pi}{2}$?

PROOF 1

(moving reference frame)

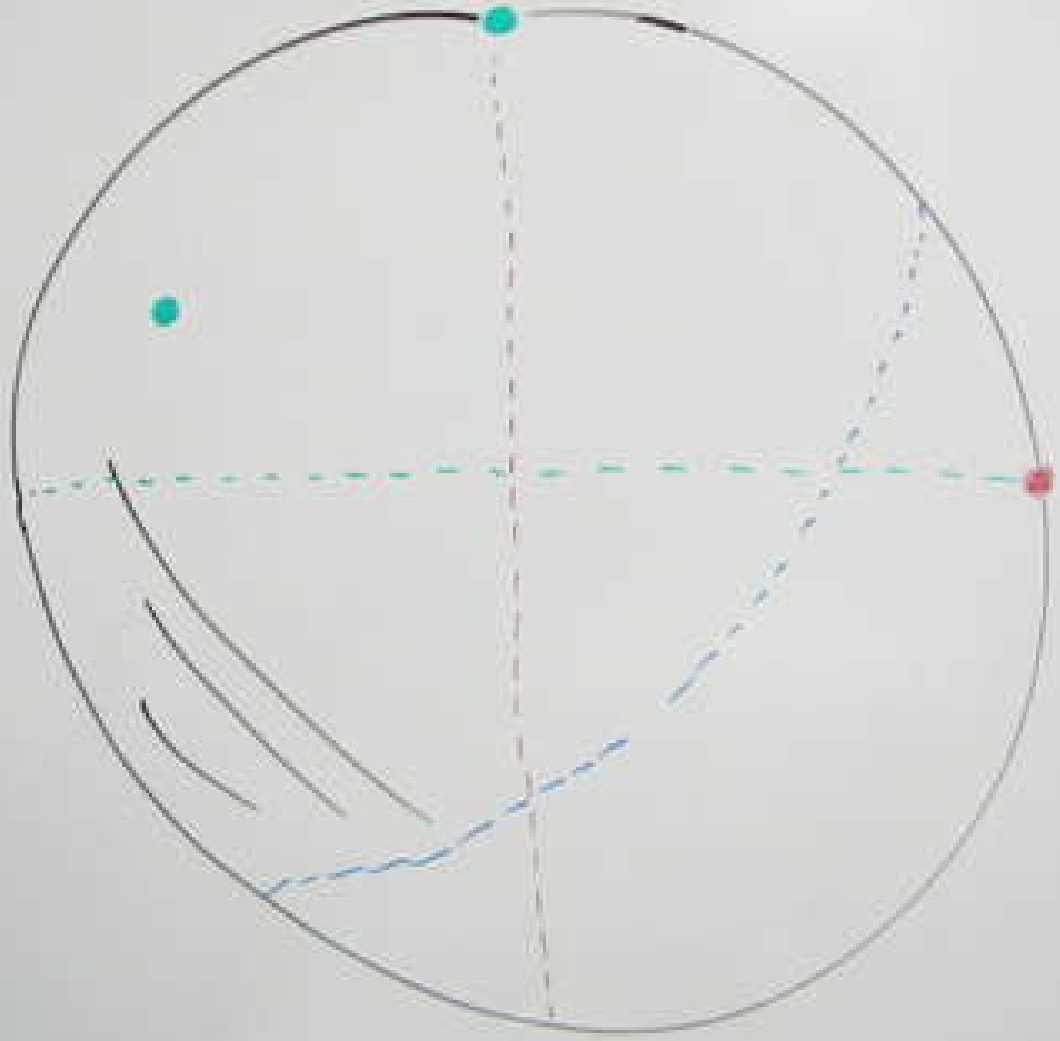


CASE I

(∞, ∞) $\theta = \frac{\pi}{2}$?

PROOF 1

(moving reference frame)

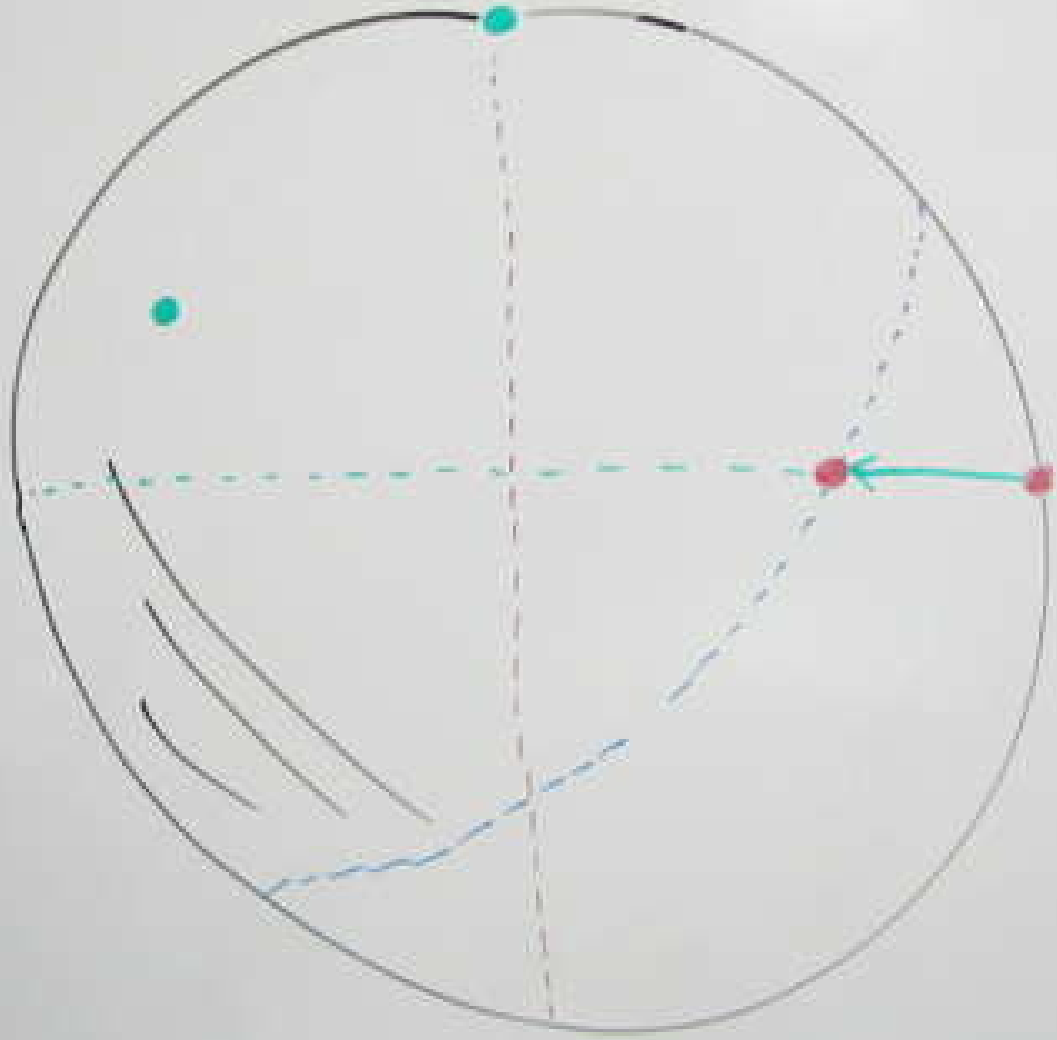


CASE I

(∞, ∞) $\theta = \frac{\pi}{2}$?

PROOF 1

(moving reference frame)

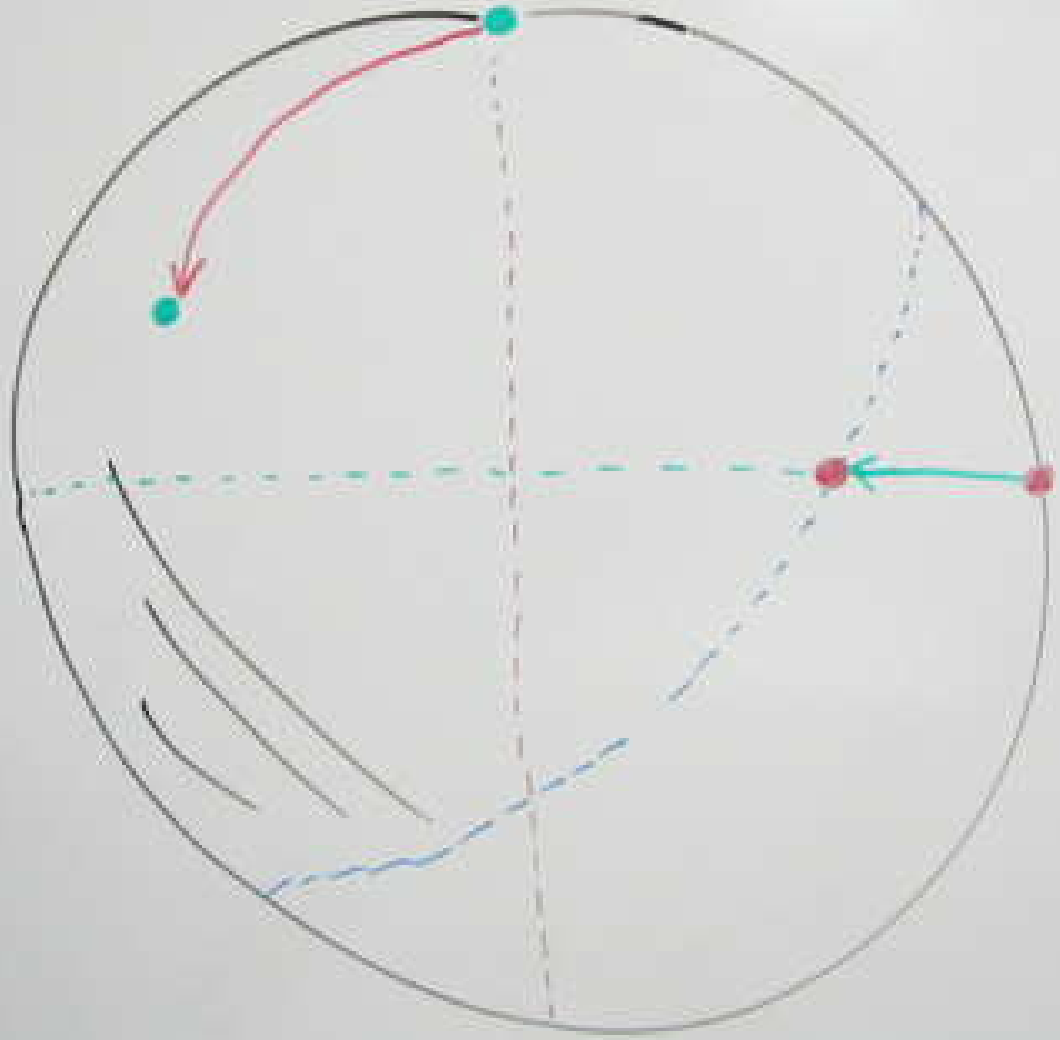


CASE I

(∞, ∞) $\theta = \frac{\pi}{2}$?

PROOF 1

(moving reference frame)



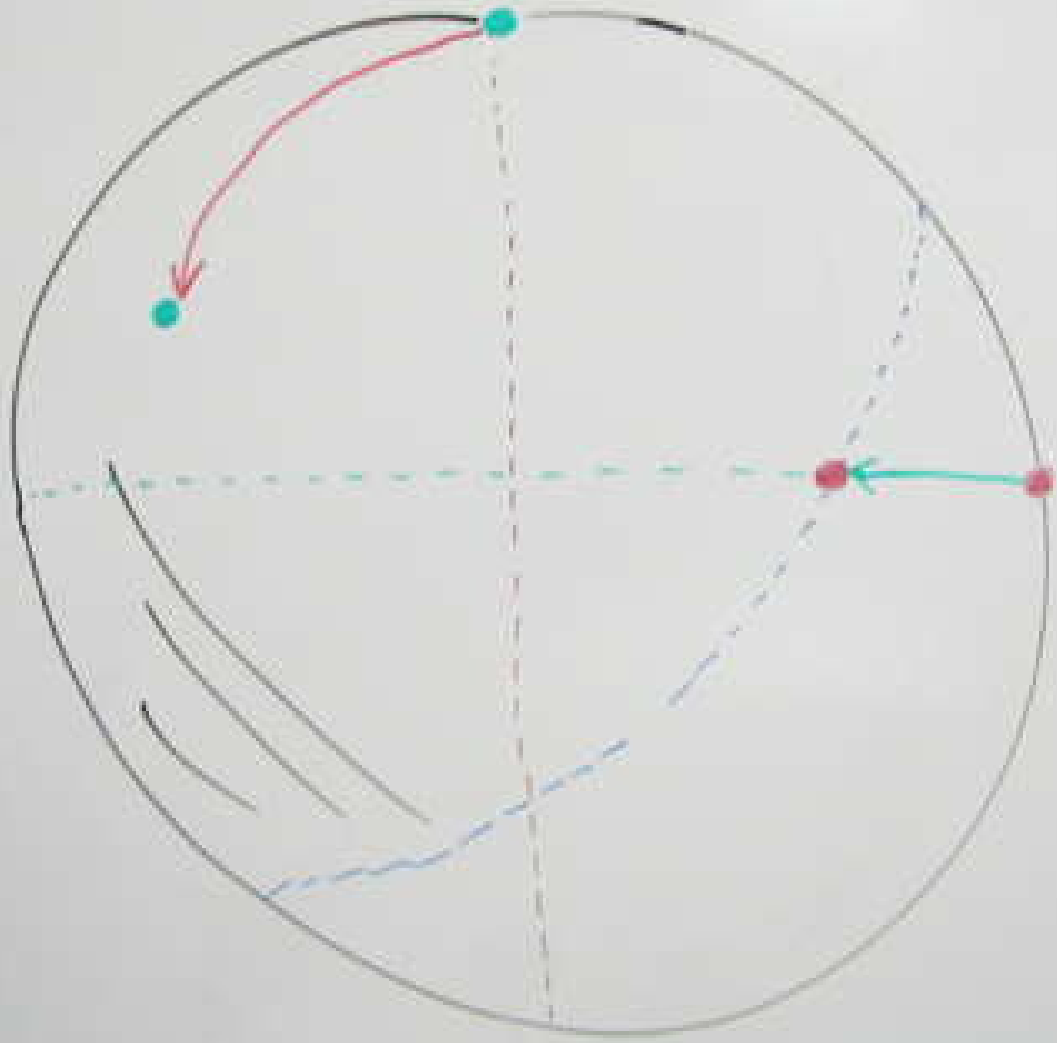
CASE I

(∞, ∞)

$\theta = \frac{\pi}{2}$ OK

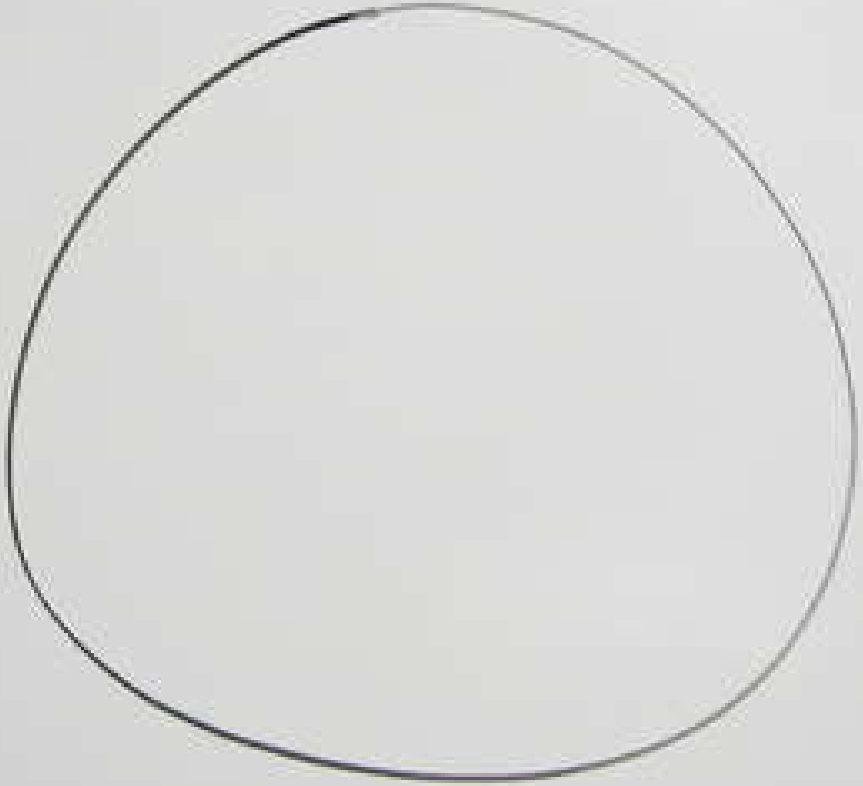
PROOF 1

(moving reference frame)



CASE I (∞, ∞)

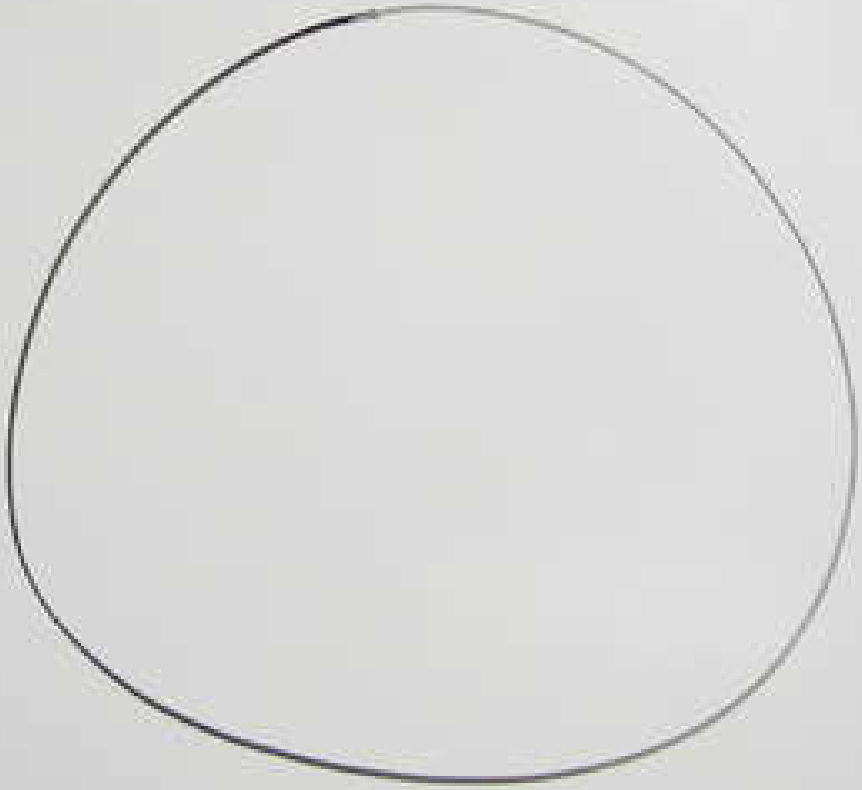
PROOF 2



CASE I (∞, ∞)

PROOF 2

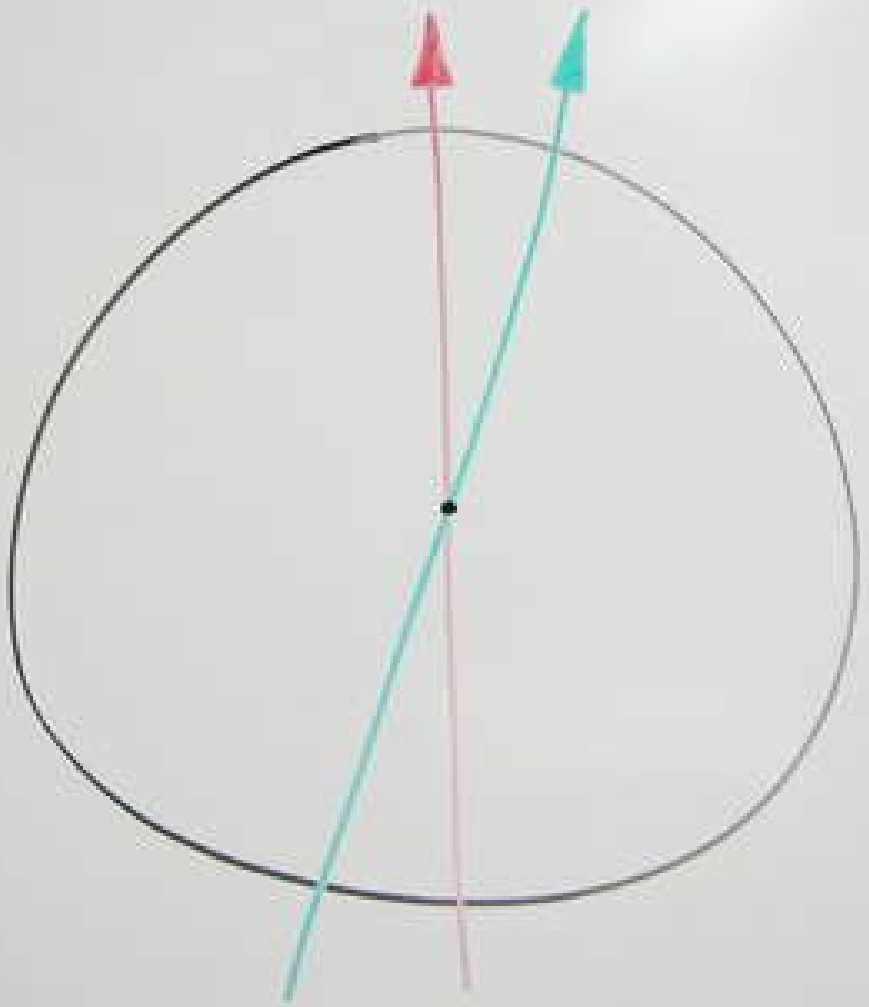
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

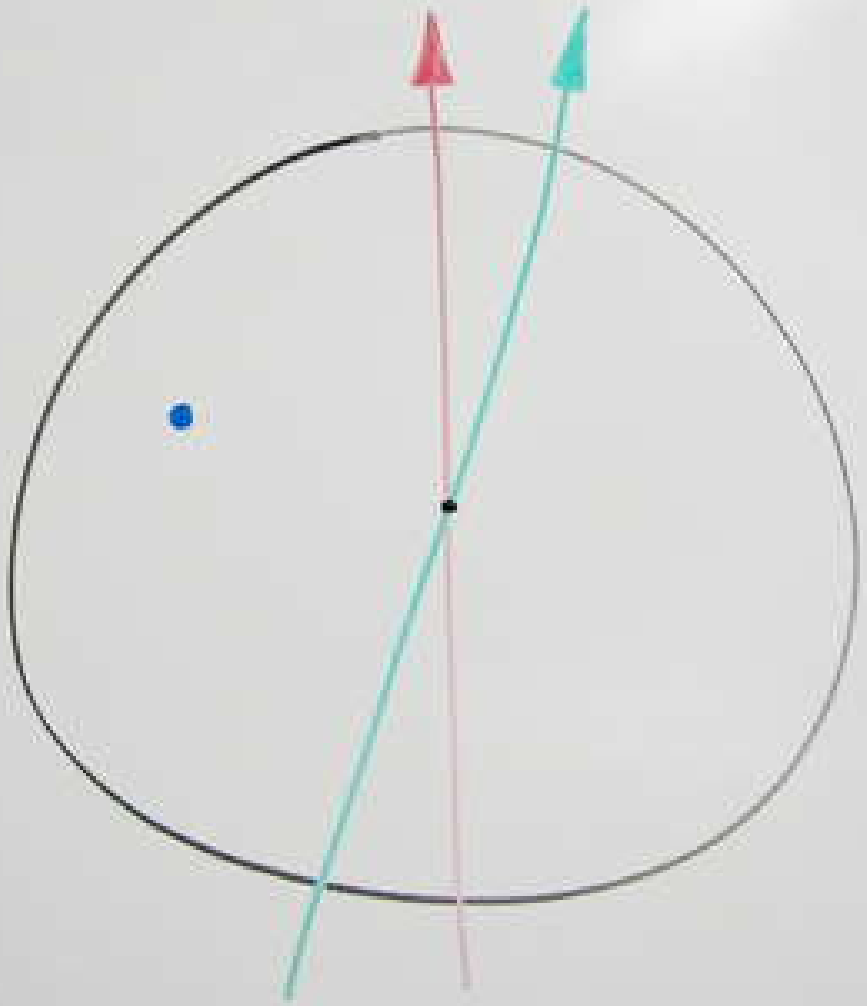
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

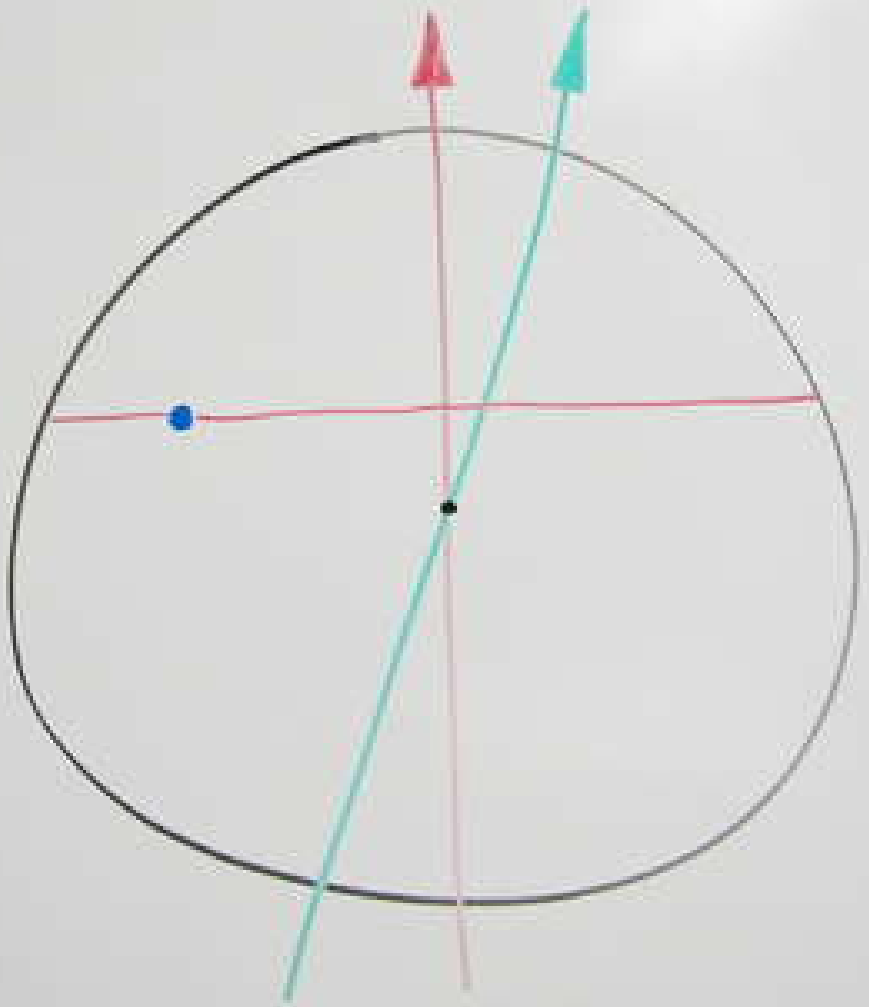
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

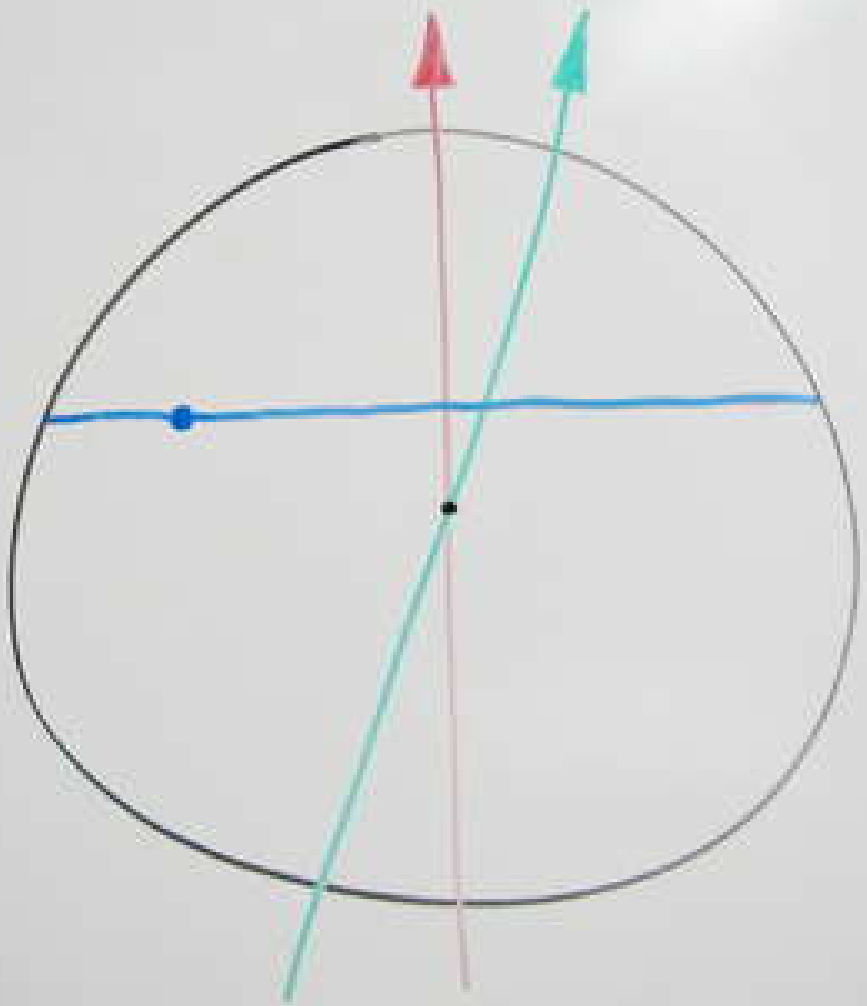
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

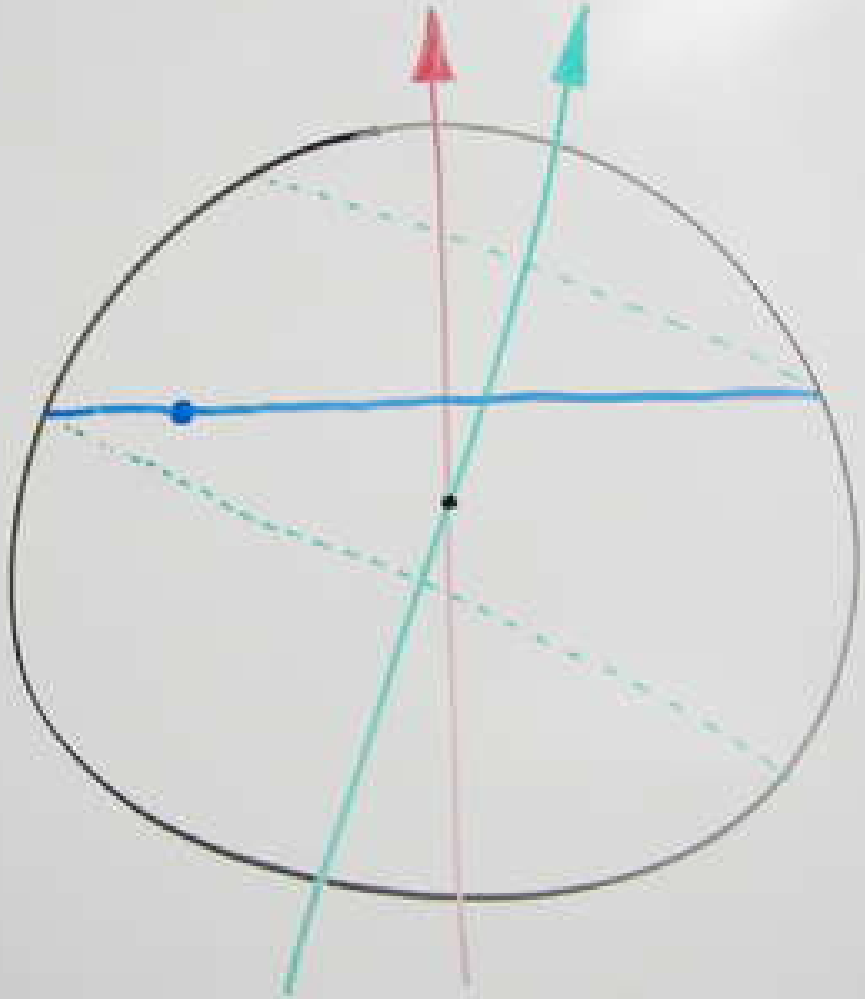
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

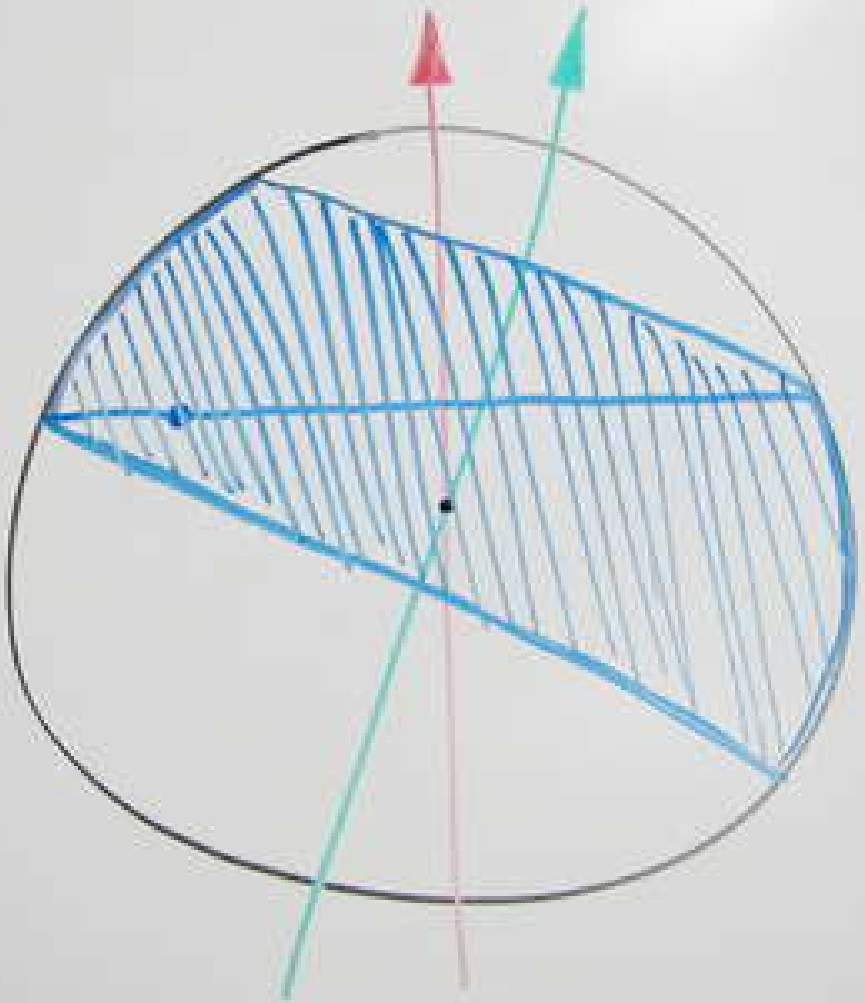
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

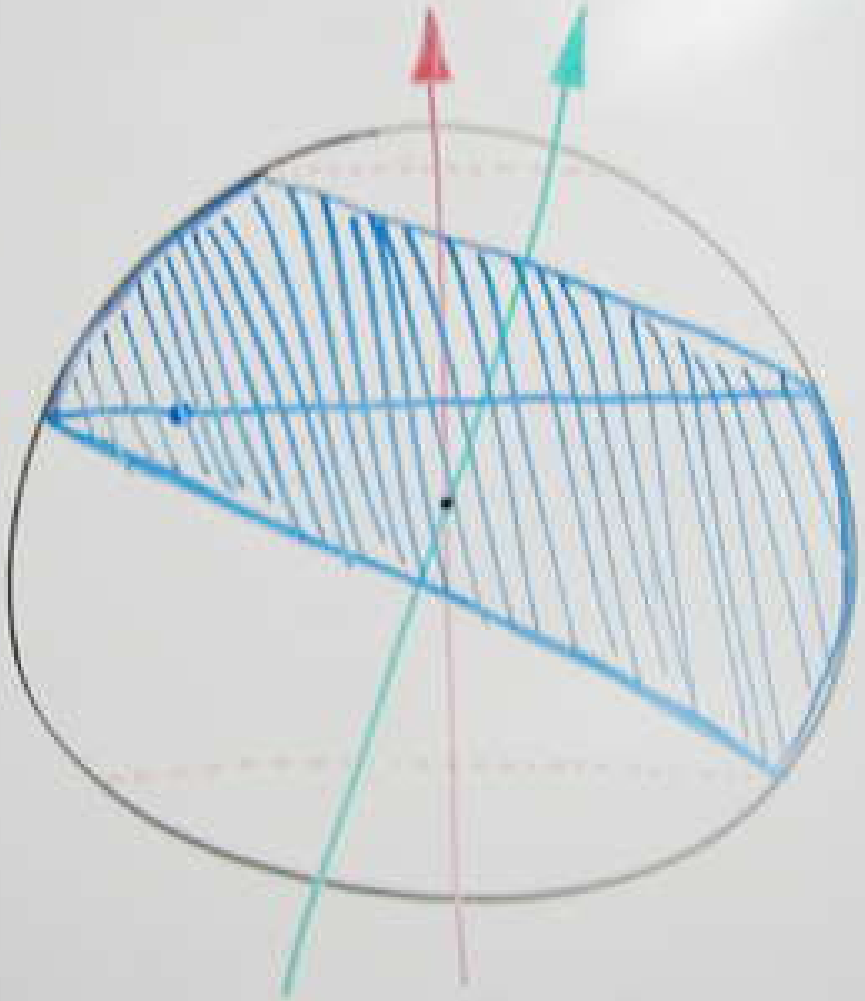
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

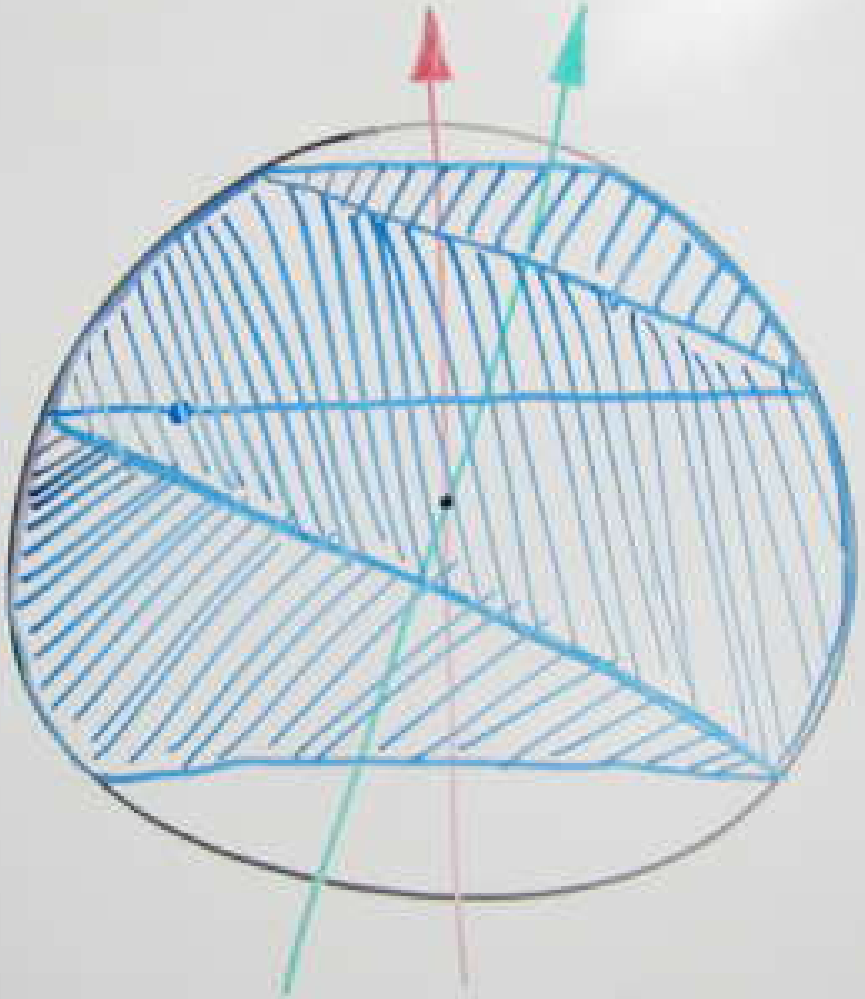
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

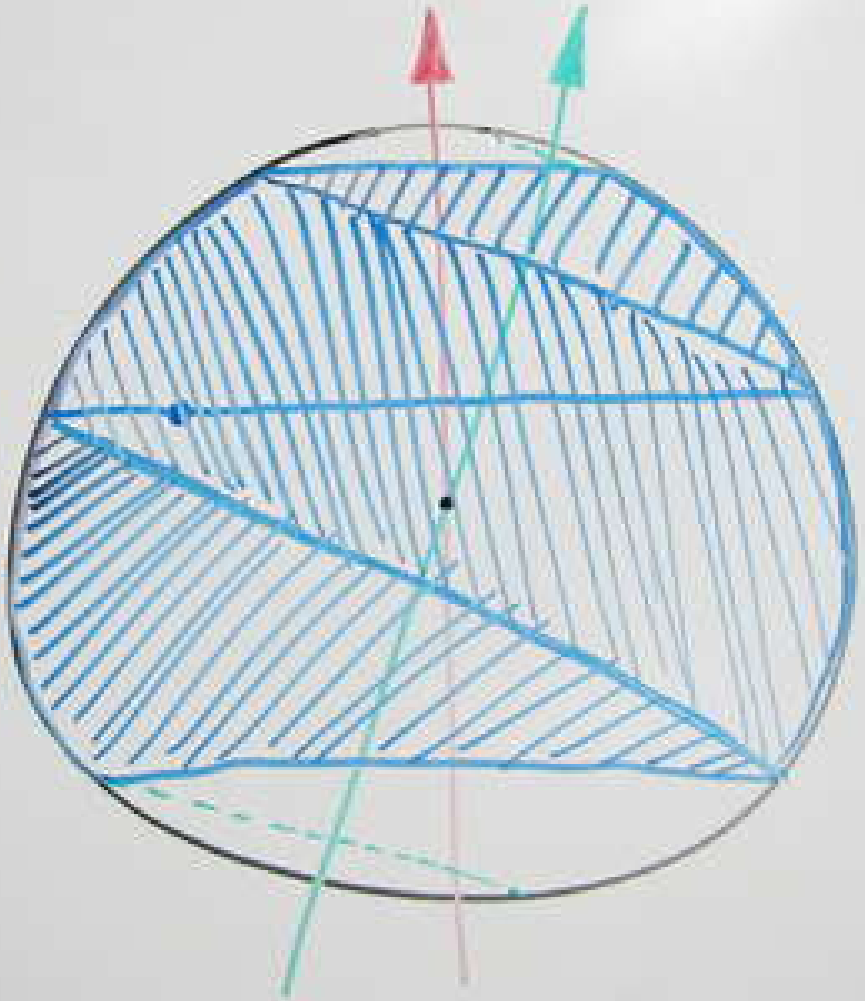
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

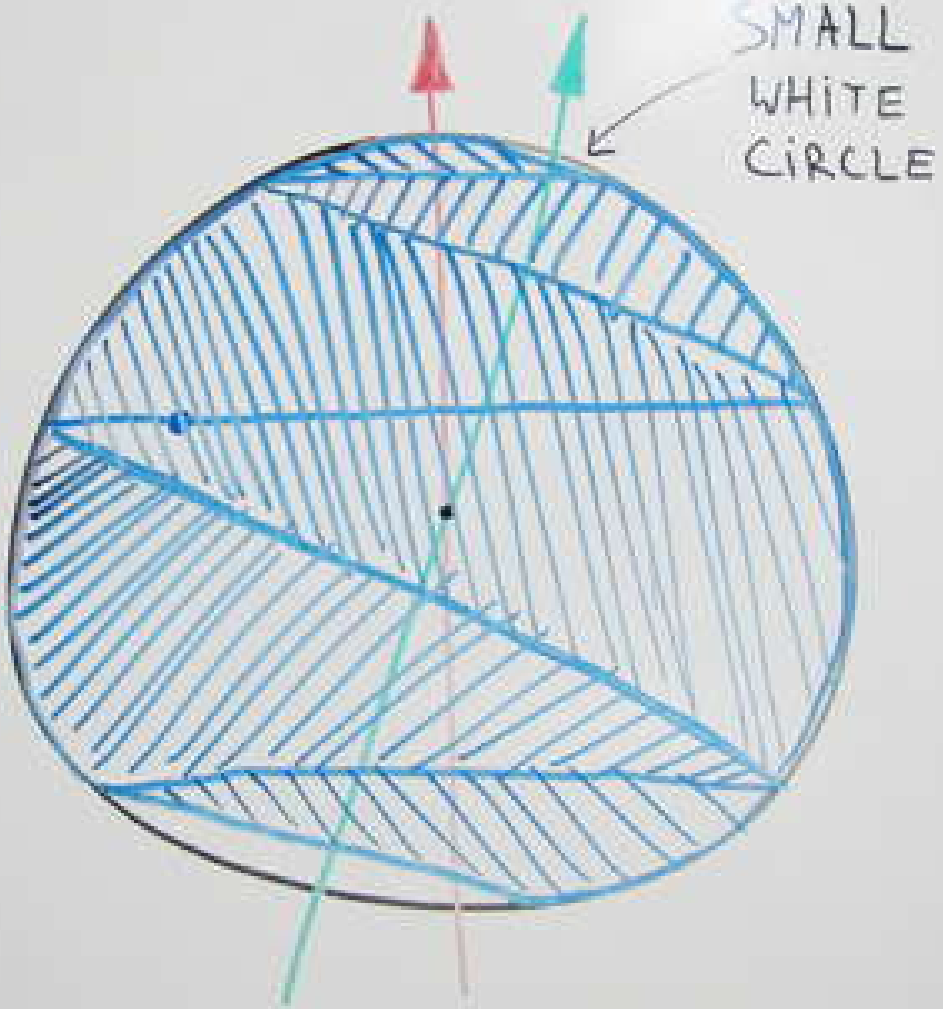
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

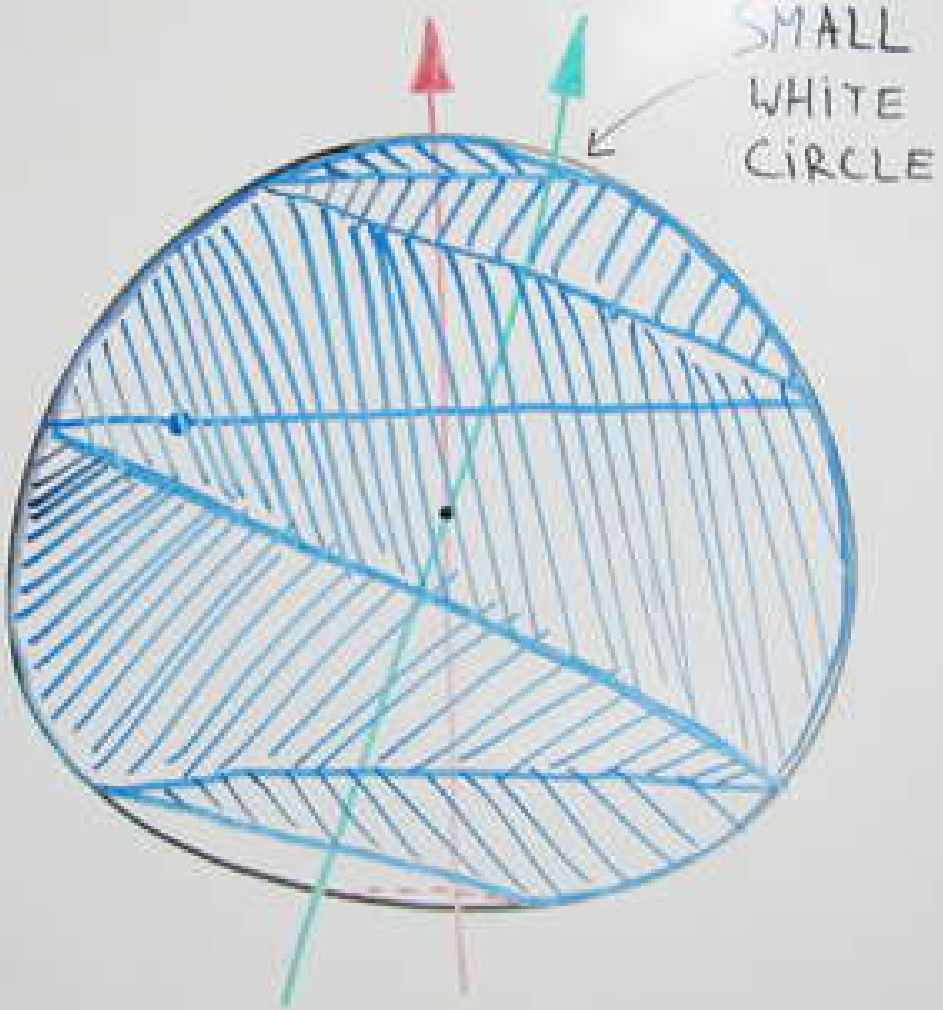
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

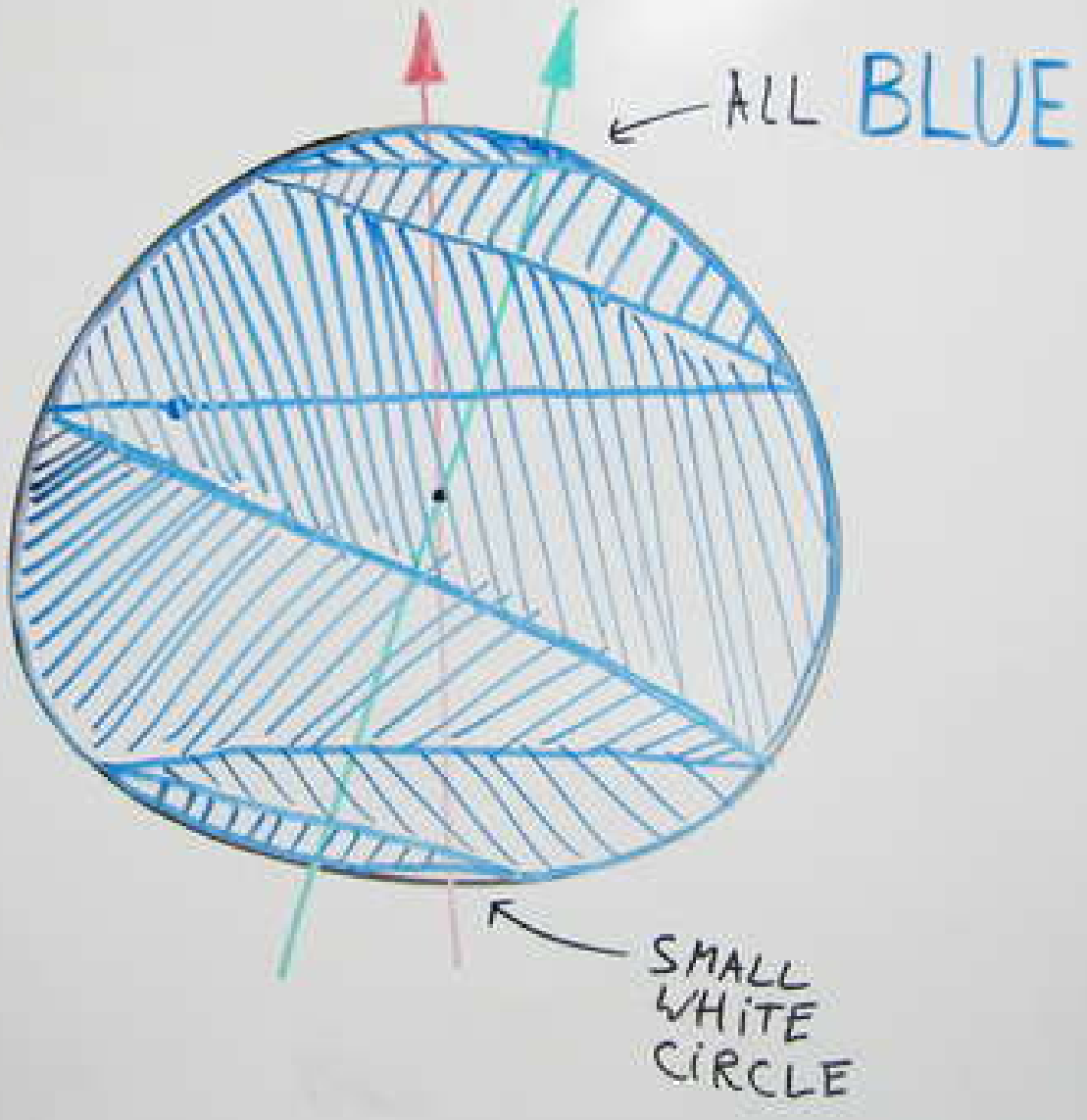
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

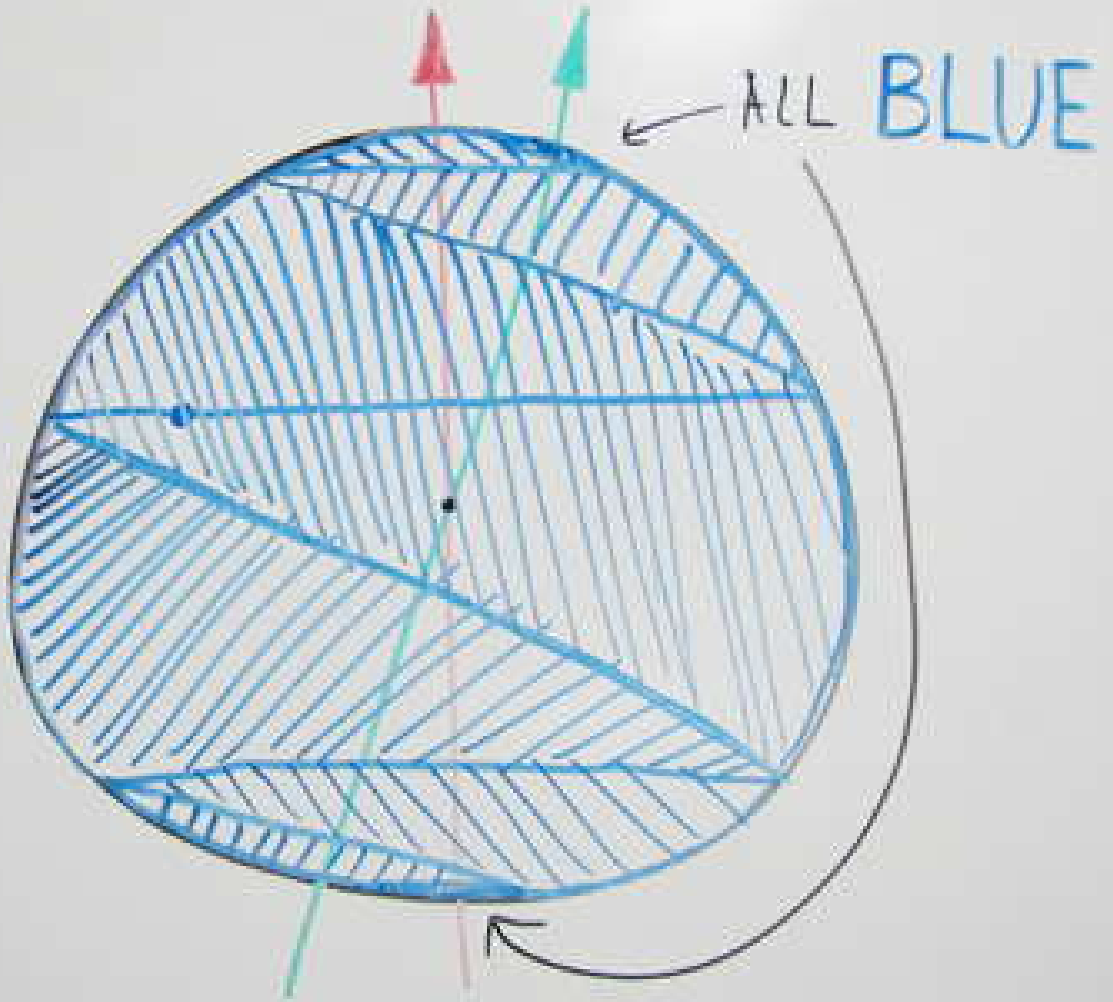
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

(fixed reference frame)



CASE I (∞, ∞)

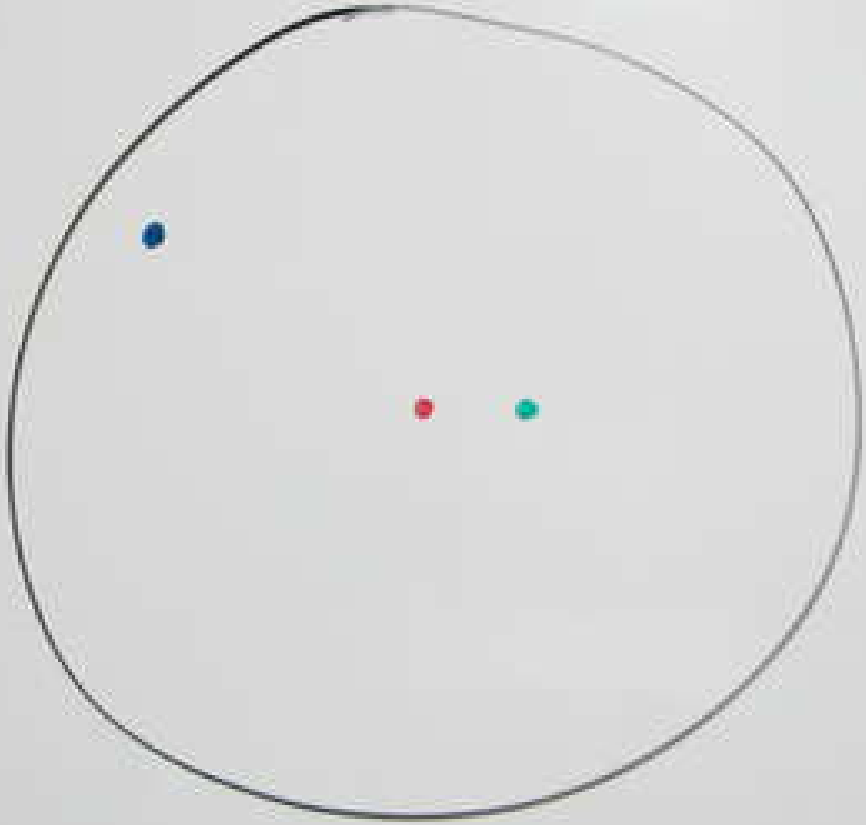
PROOF 2

(fixed reference frame)

CASE I (∞, ∞)

PROOF 2

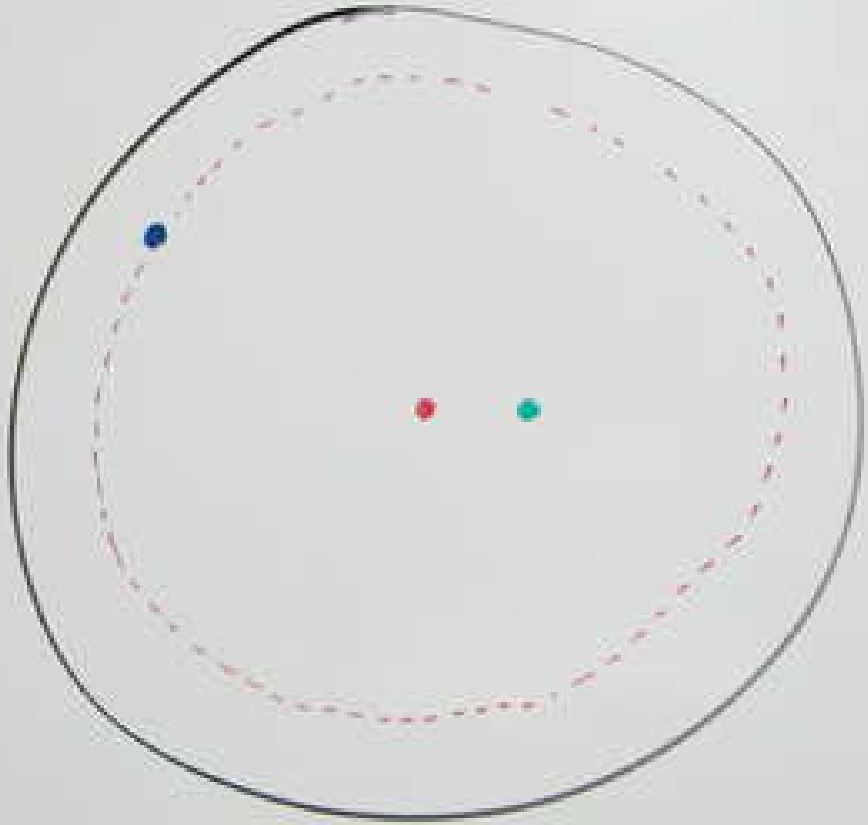
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

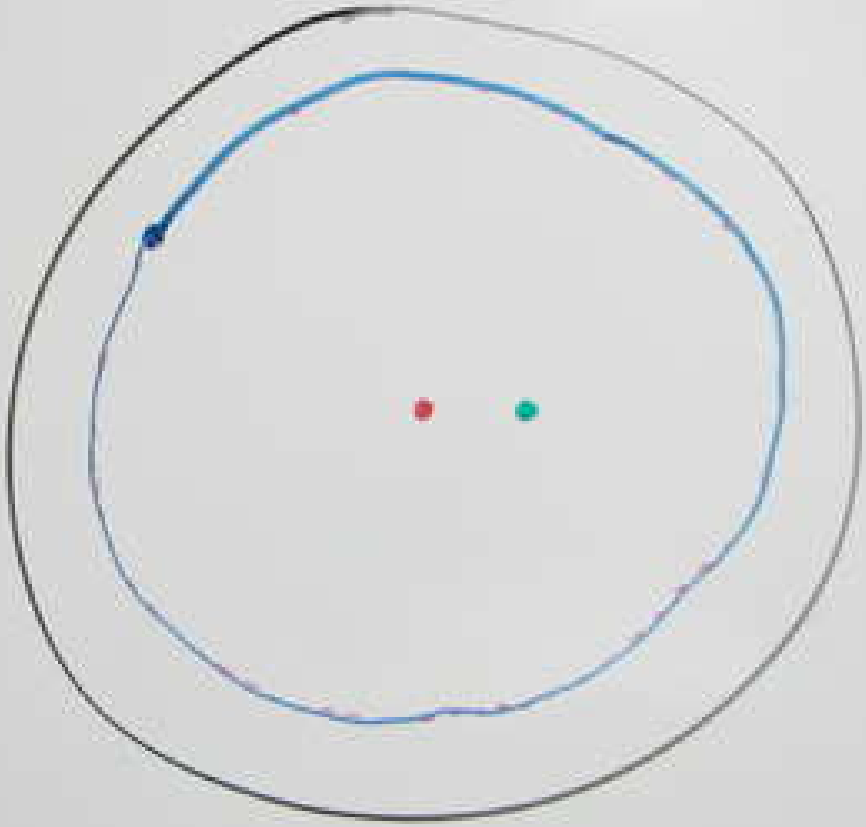
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

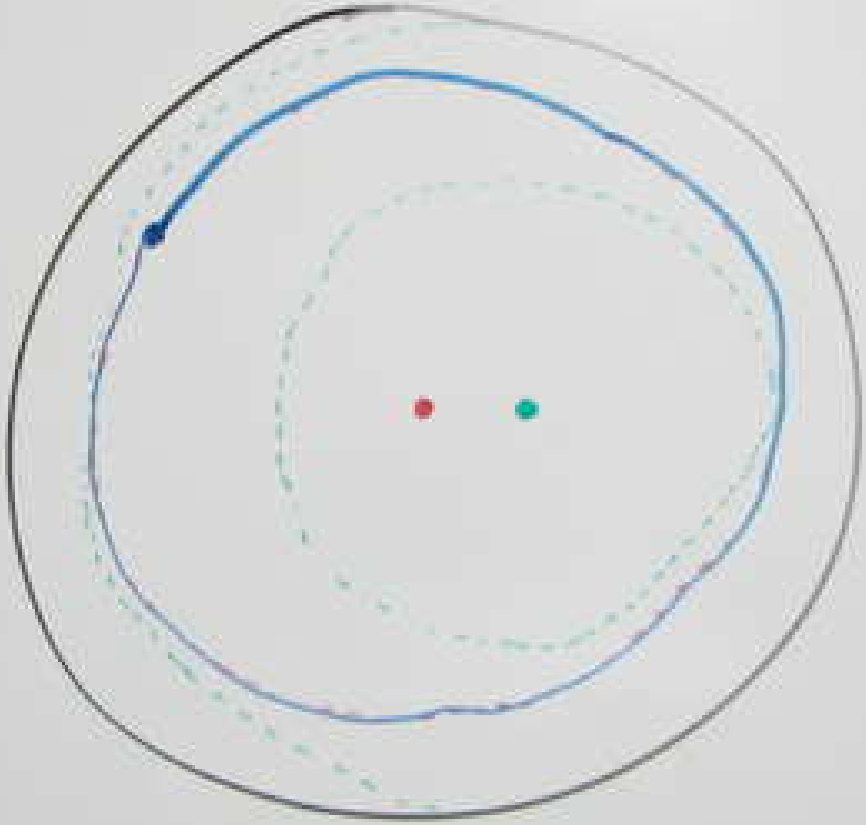
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

(fixed reference frame)

CASE I (∞, ∞)

PROOF 2

(fixed reference frame)

CASE I (∞, ∞)

PROOF 2

(fixed reference frame)

CASE I (∞, ∞)

PROOF 2

(fixed reference frame)

CASE I (∞, ∞)

PROOF 2

(fixed reference frame)

CASE I (∞, ∞)

PROOF 2

(fixed reference frame)

CASE I (∞, ∞)

PROOF 2

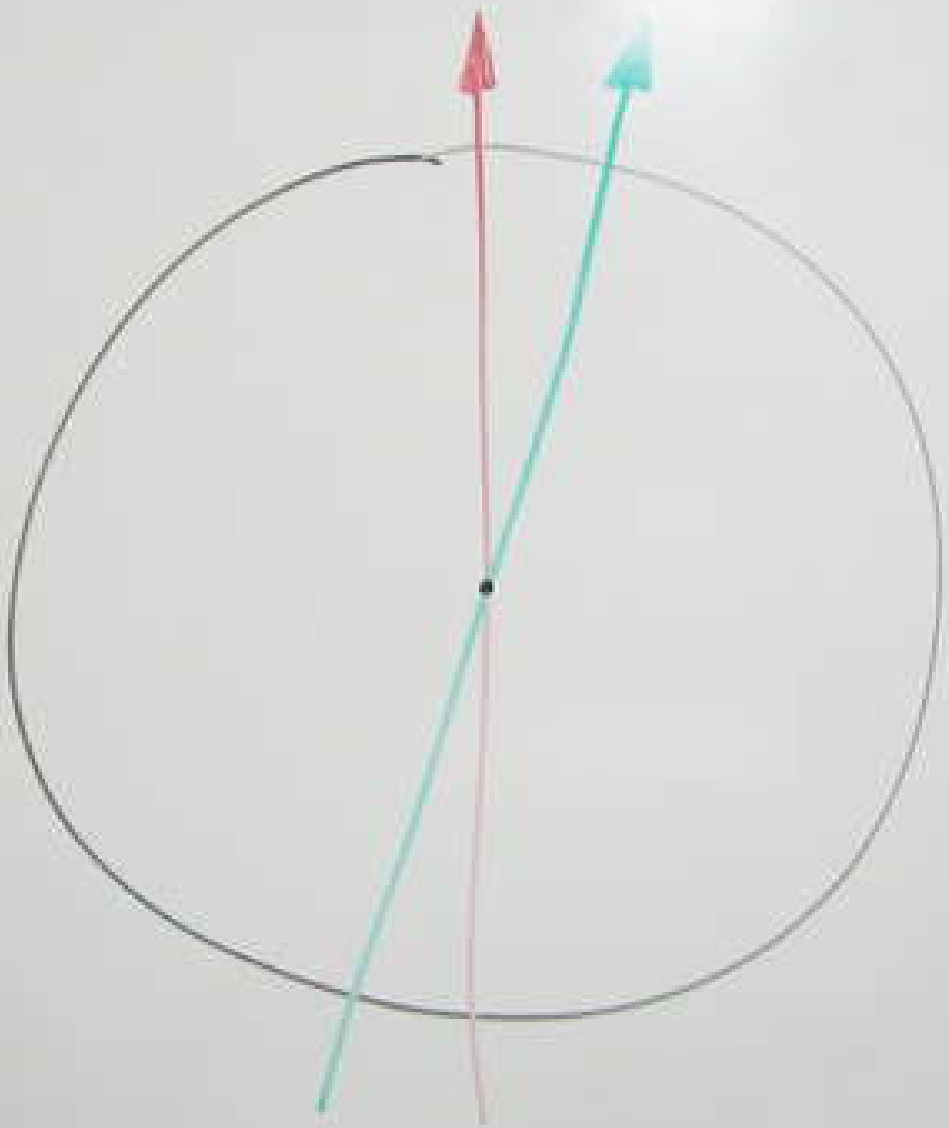
(fixed reference frame)

HOW LONG IT TAKES?

CASE I (∞, ∞)

PROOF 2

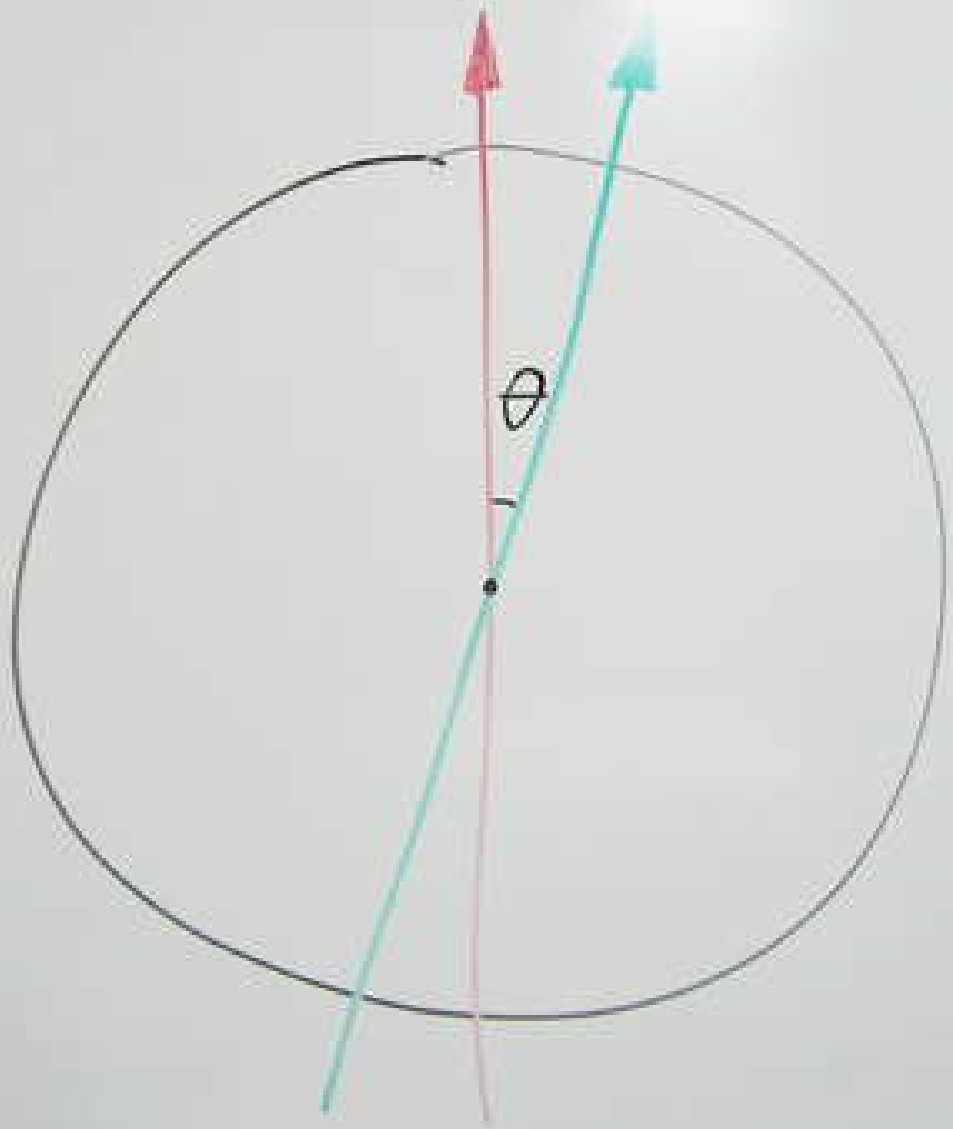
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

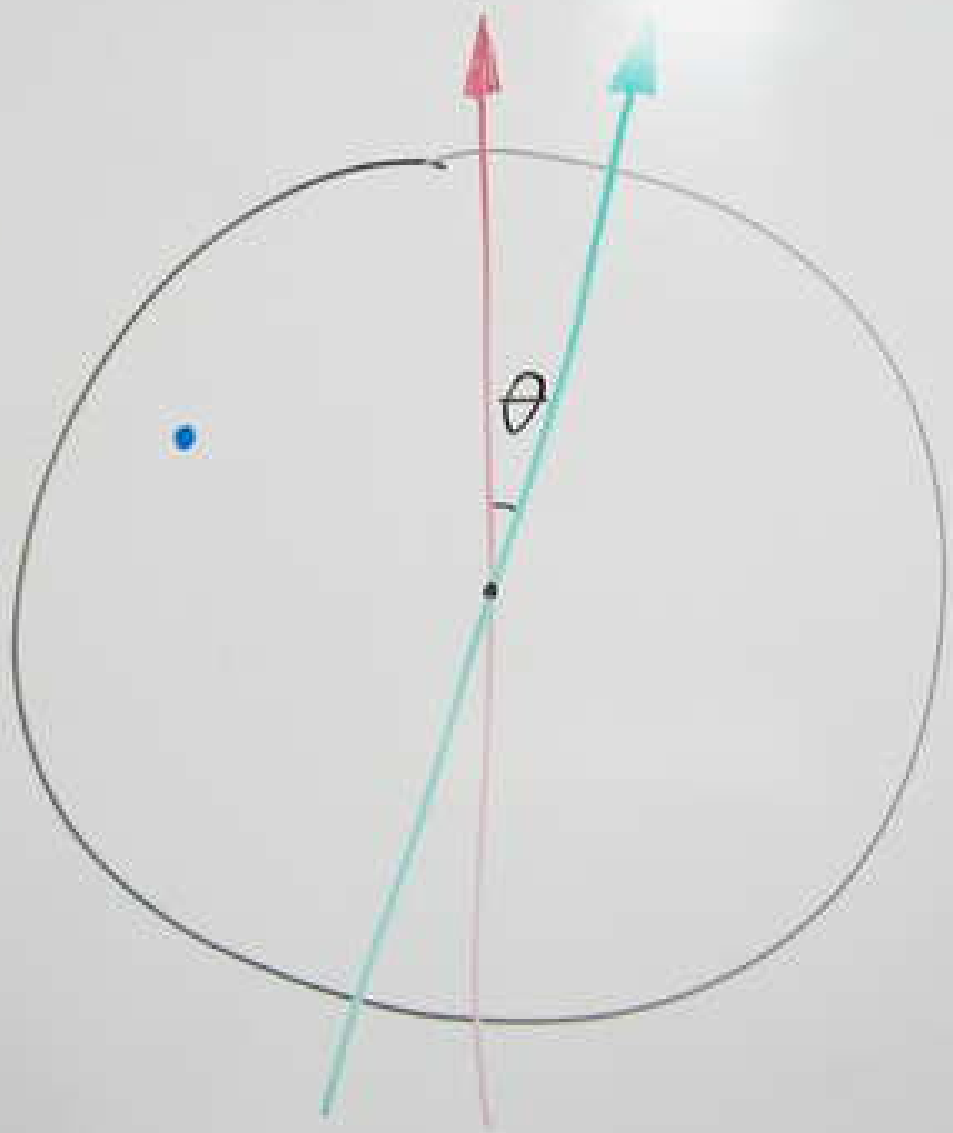
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

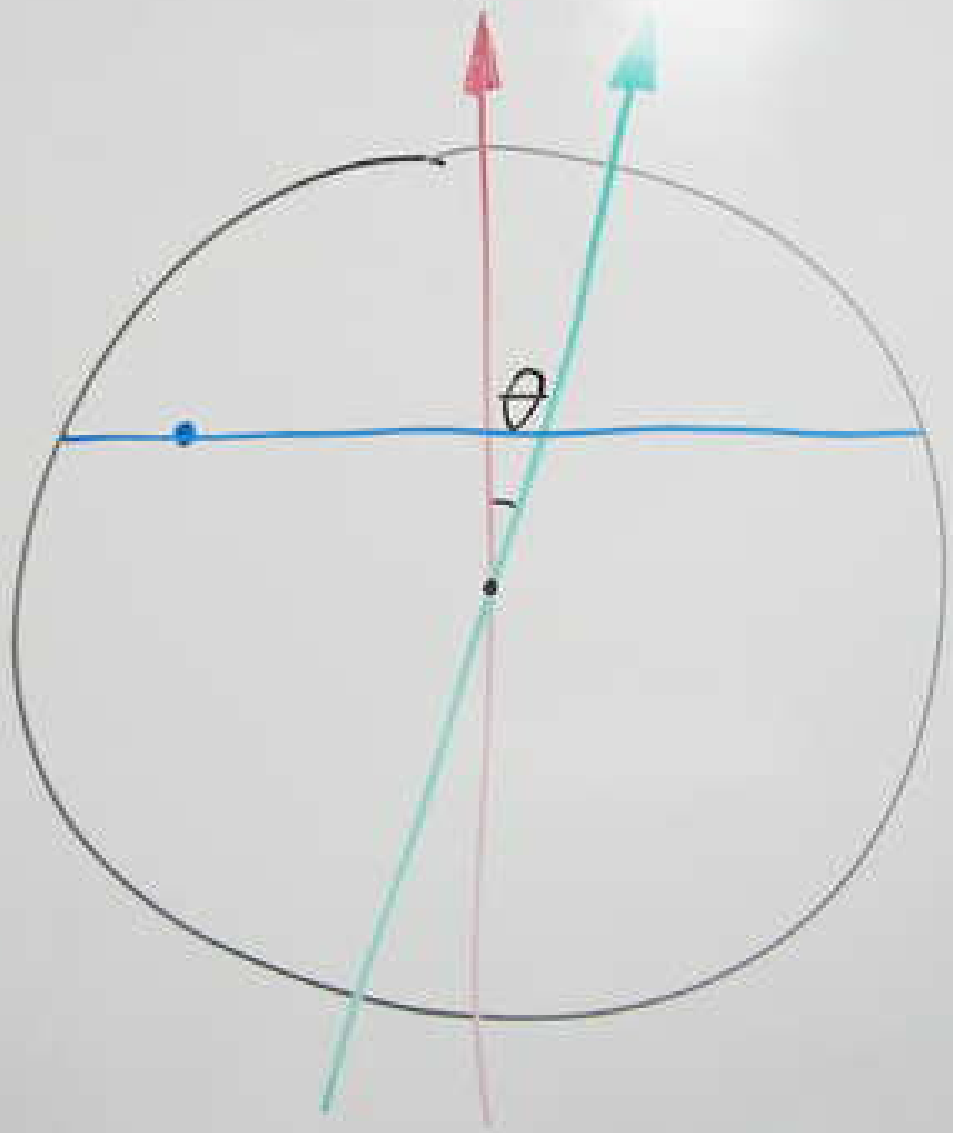
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

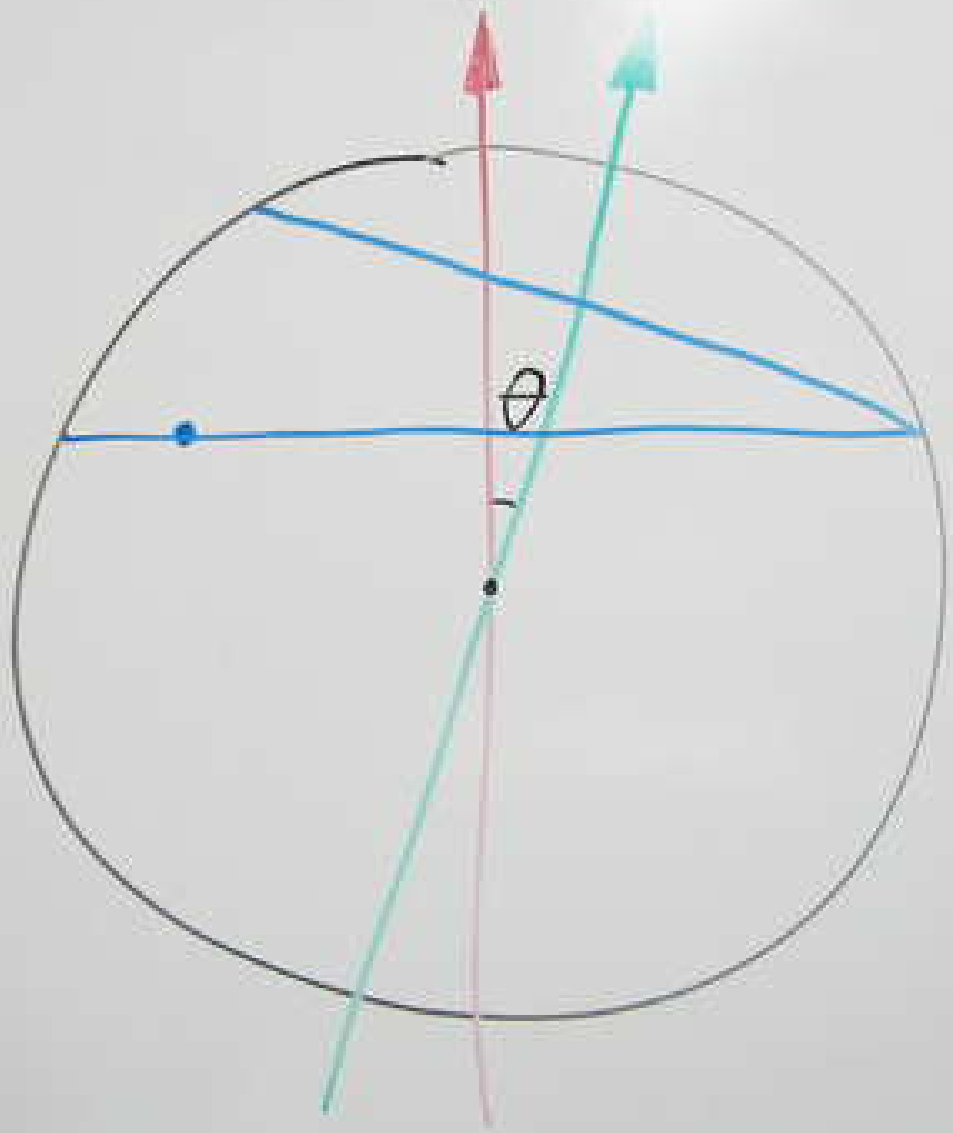
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

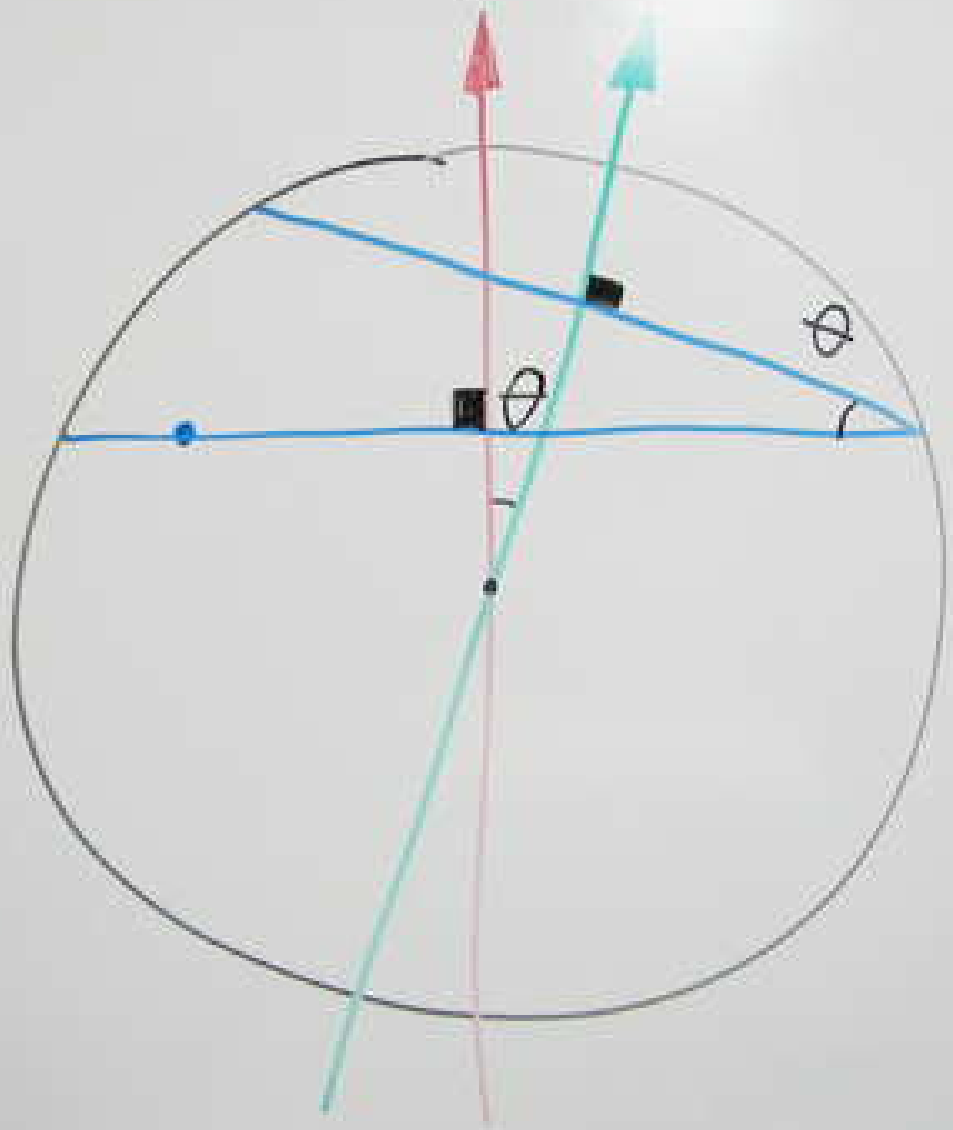
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

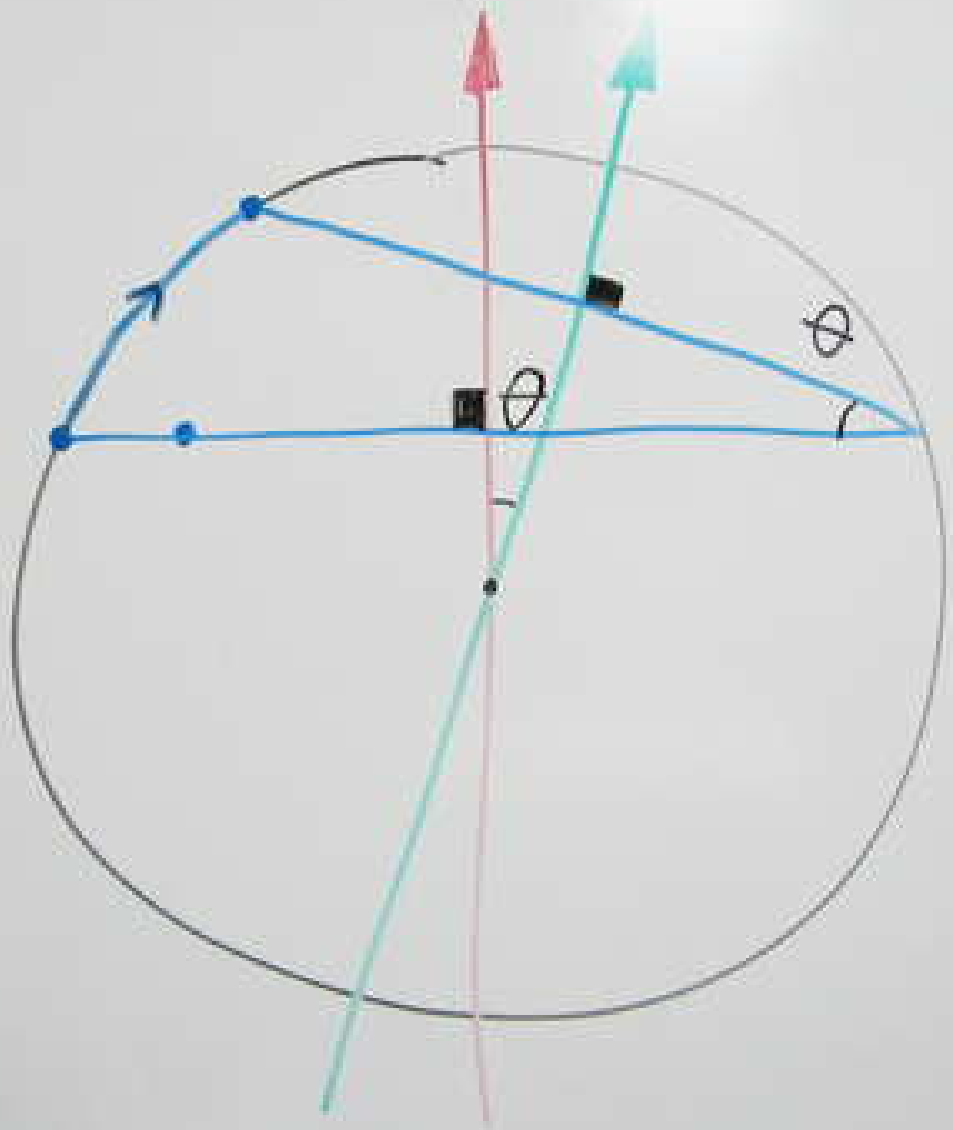
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

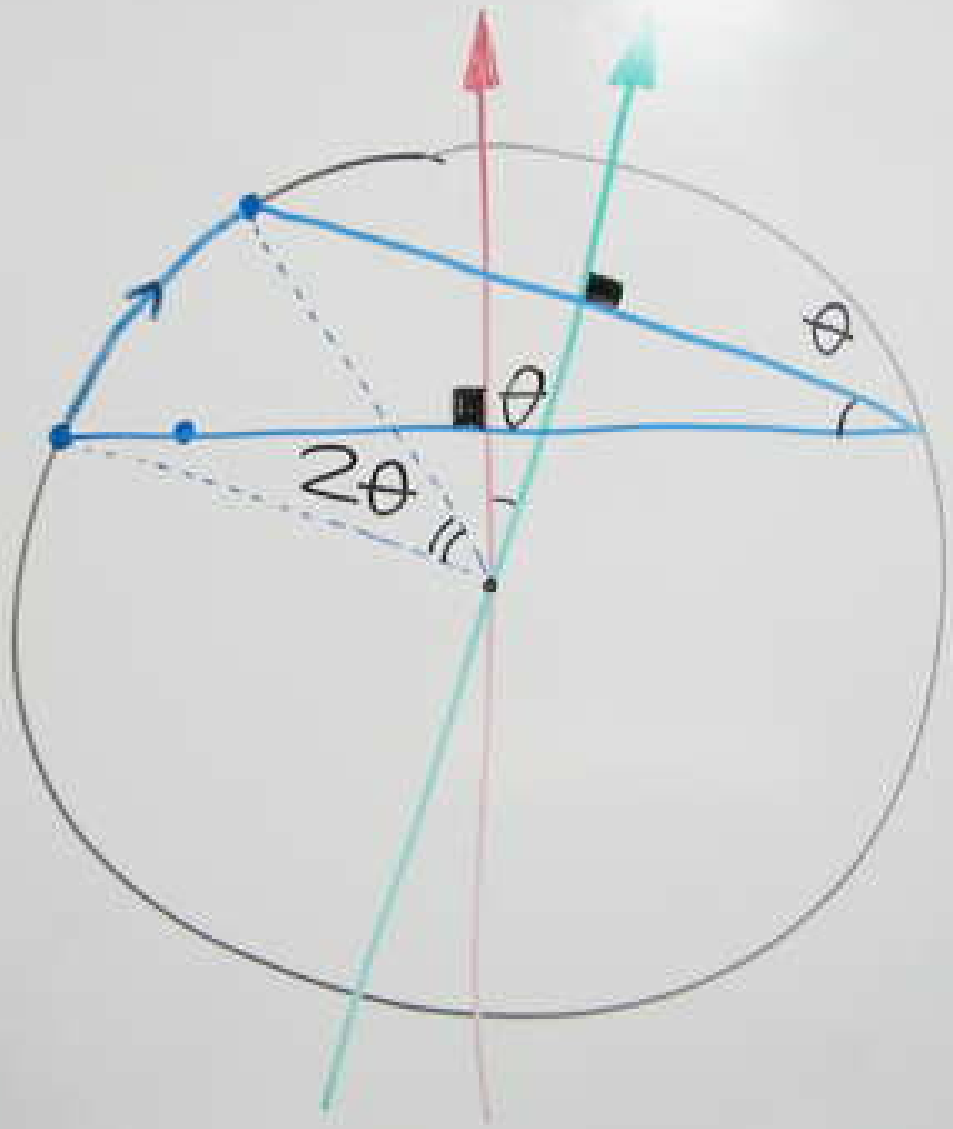
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

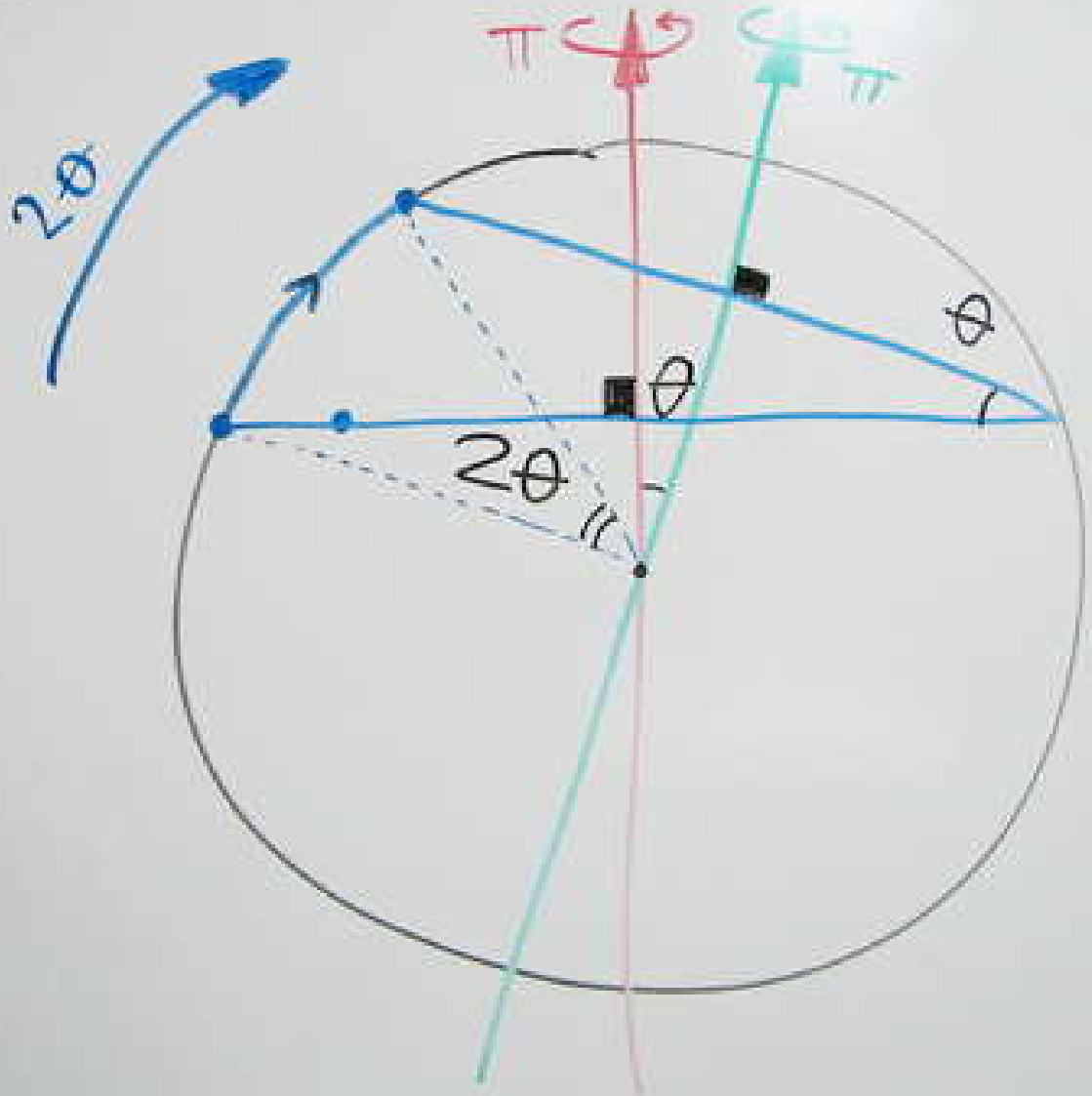
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

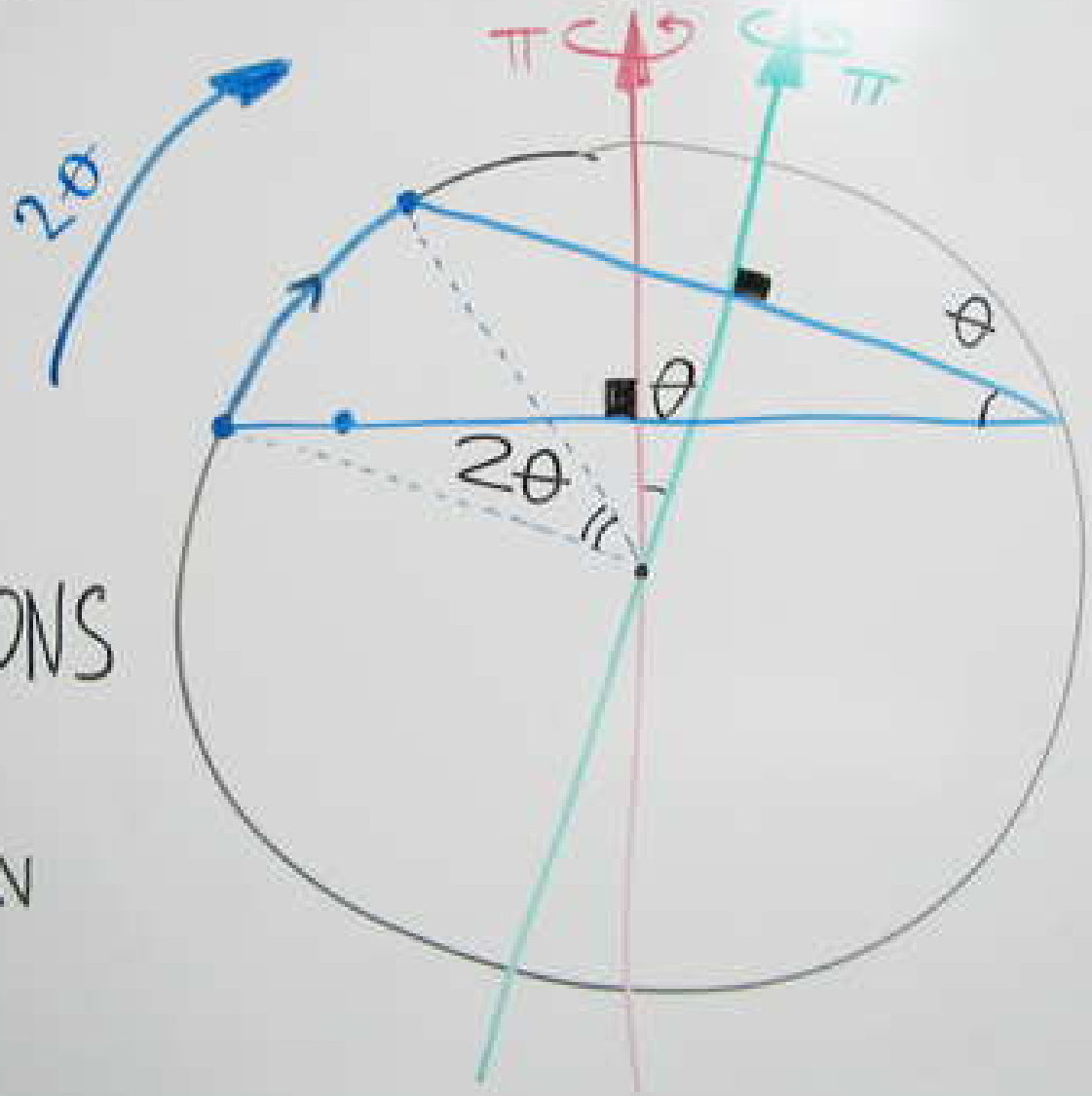
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

(fixed reference frame)



2 REFLECTIONS

III

ROTATION

CASE I (∞, ∞)

PROOF 2

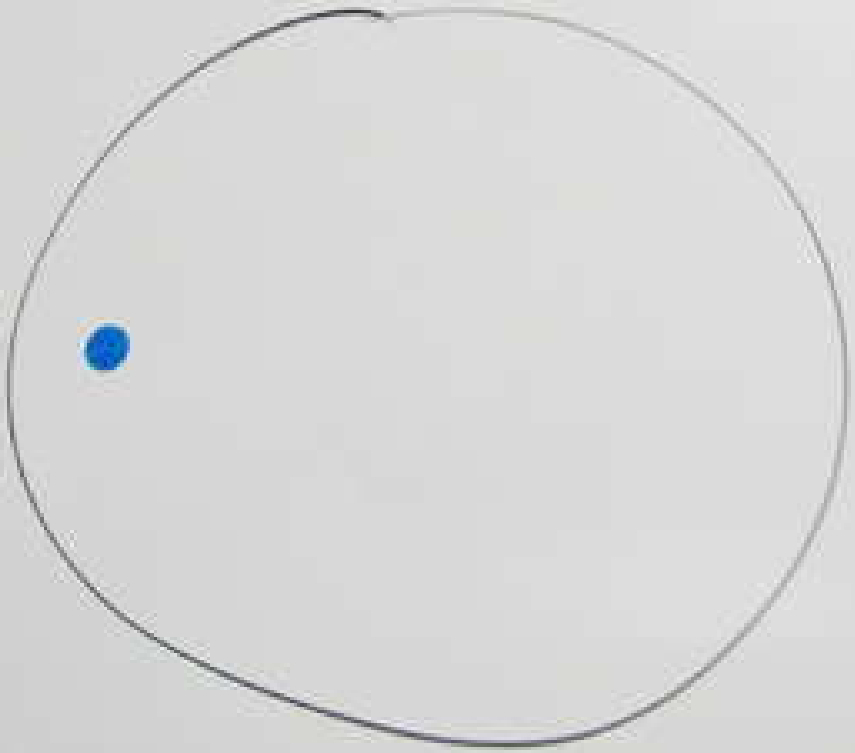
(fixed reference frame)

THAT'S NOT ALL!

CASE I (∞, ∞)

PROOF 2

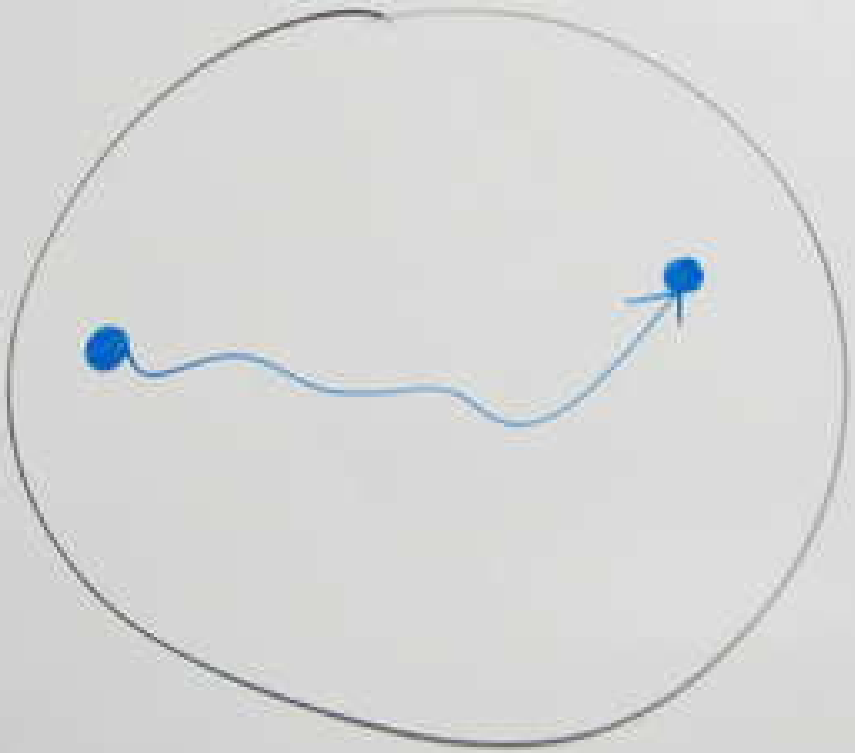
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

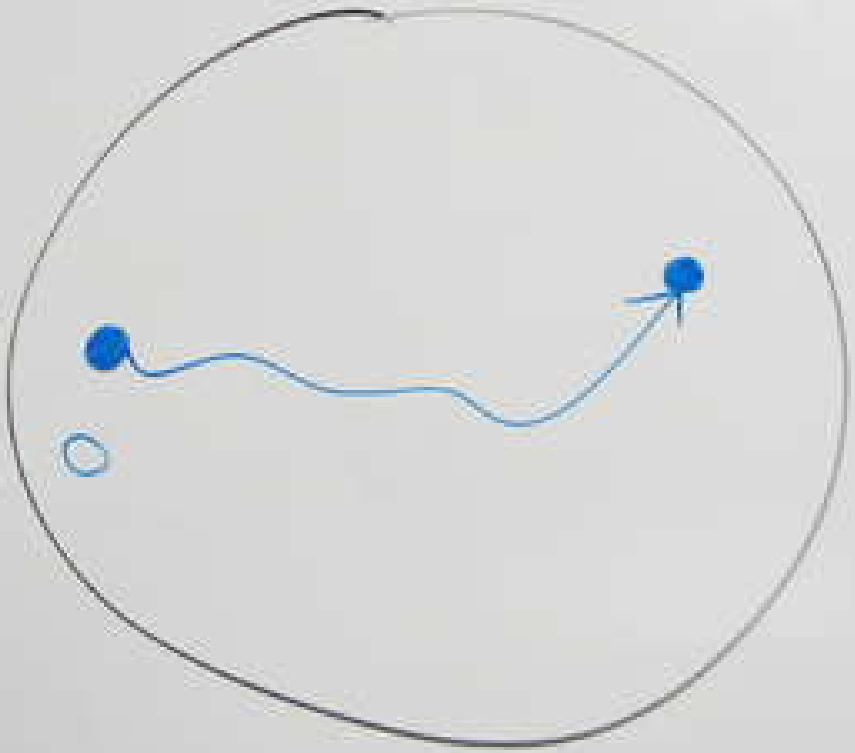
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

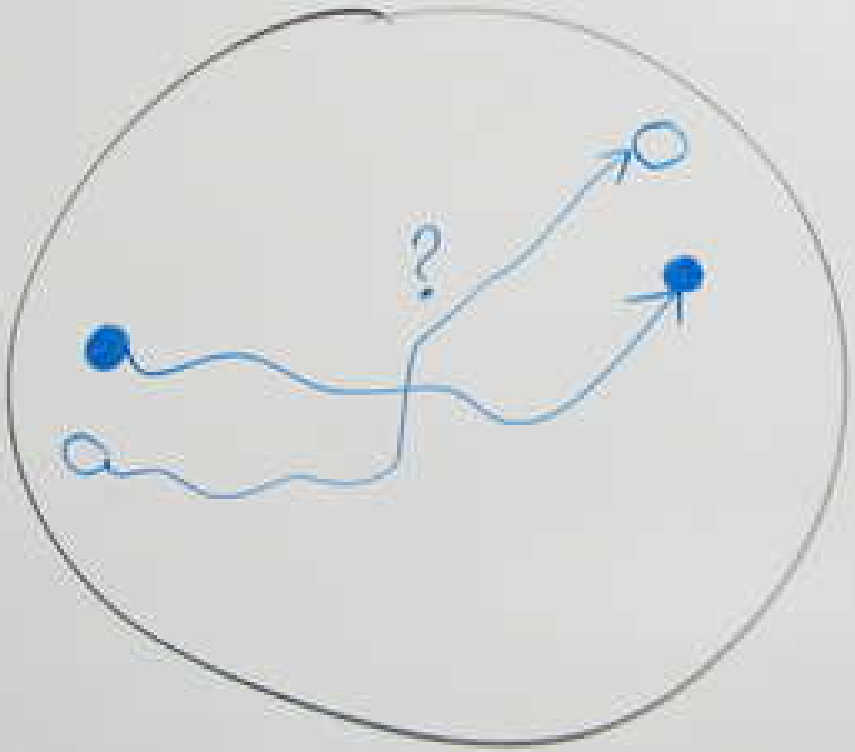
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

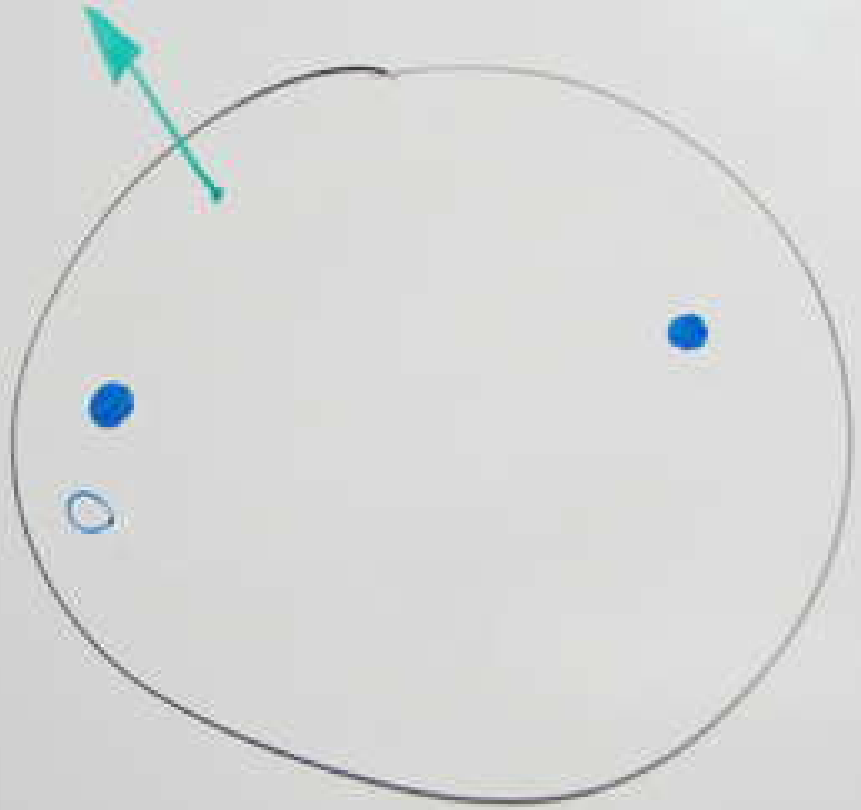
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

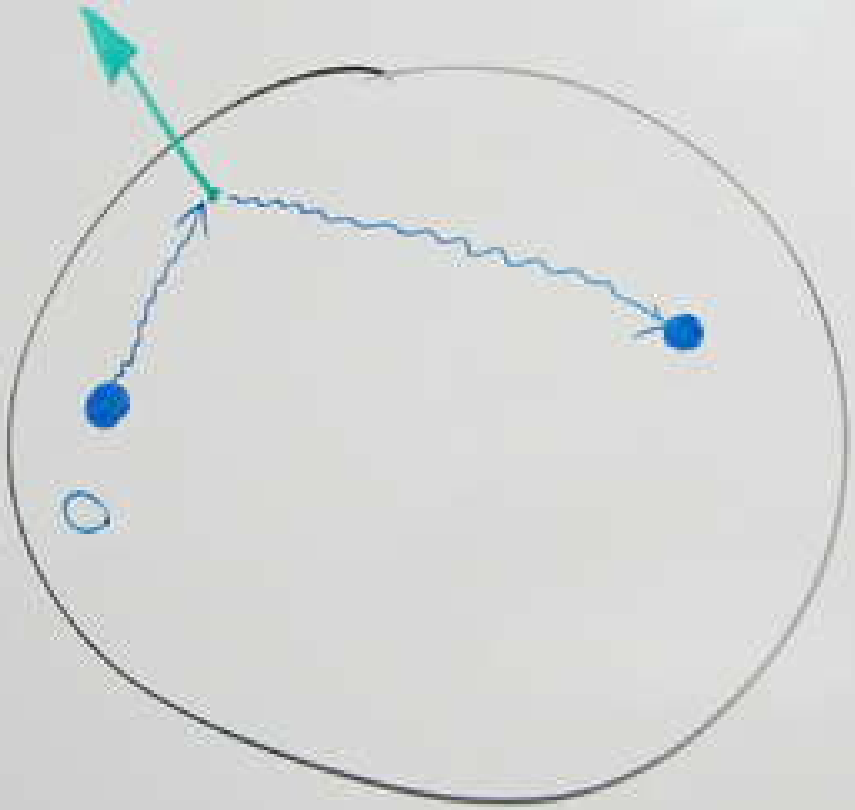
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

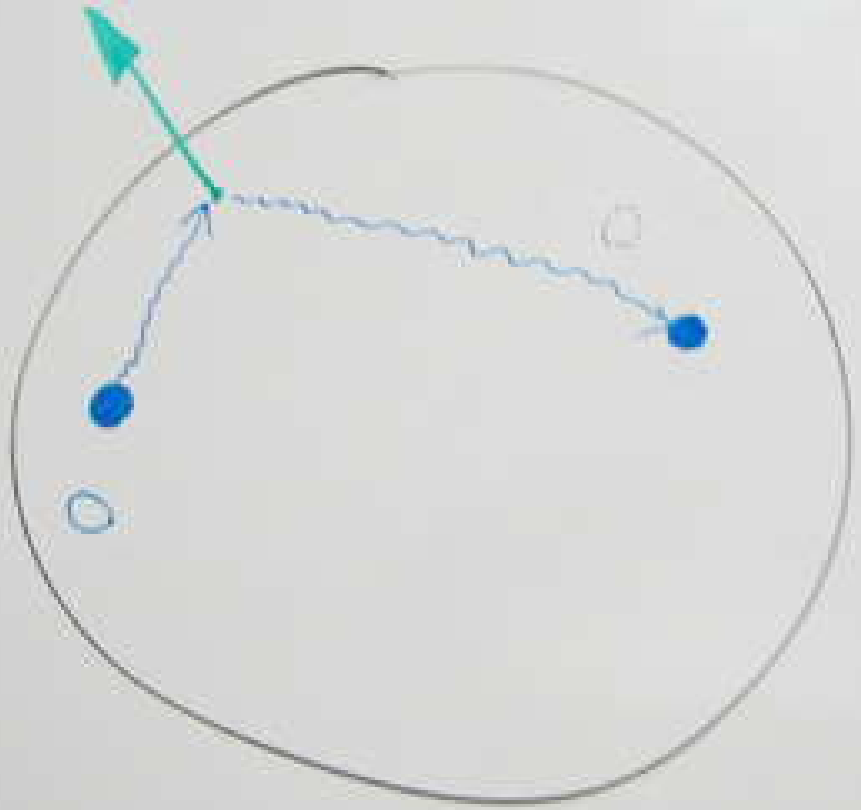
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

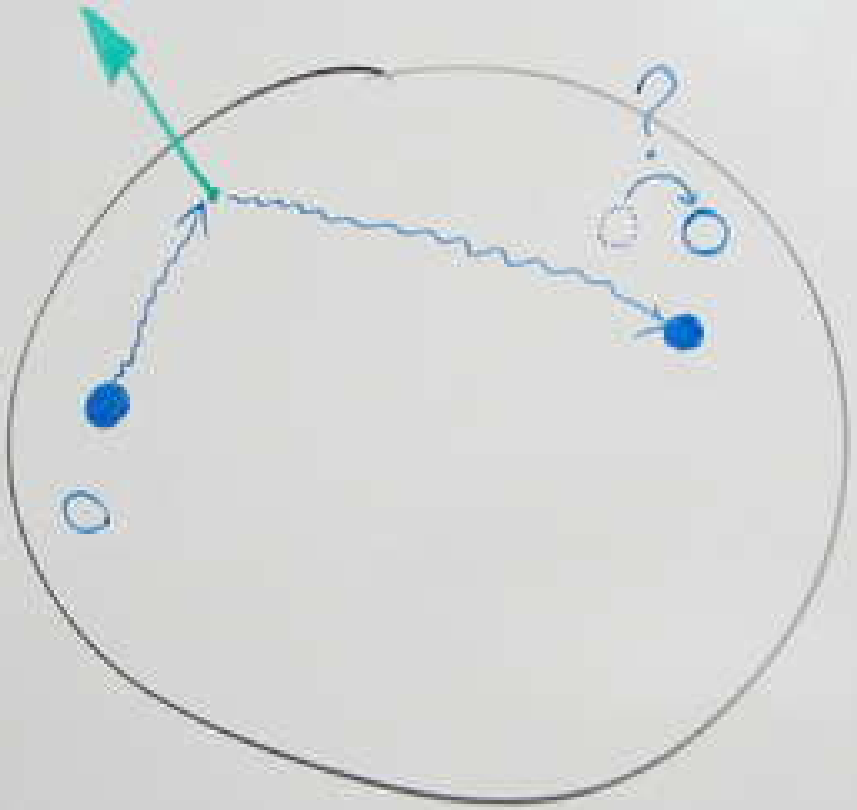
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

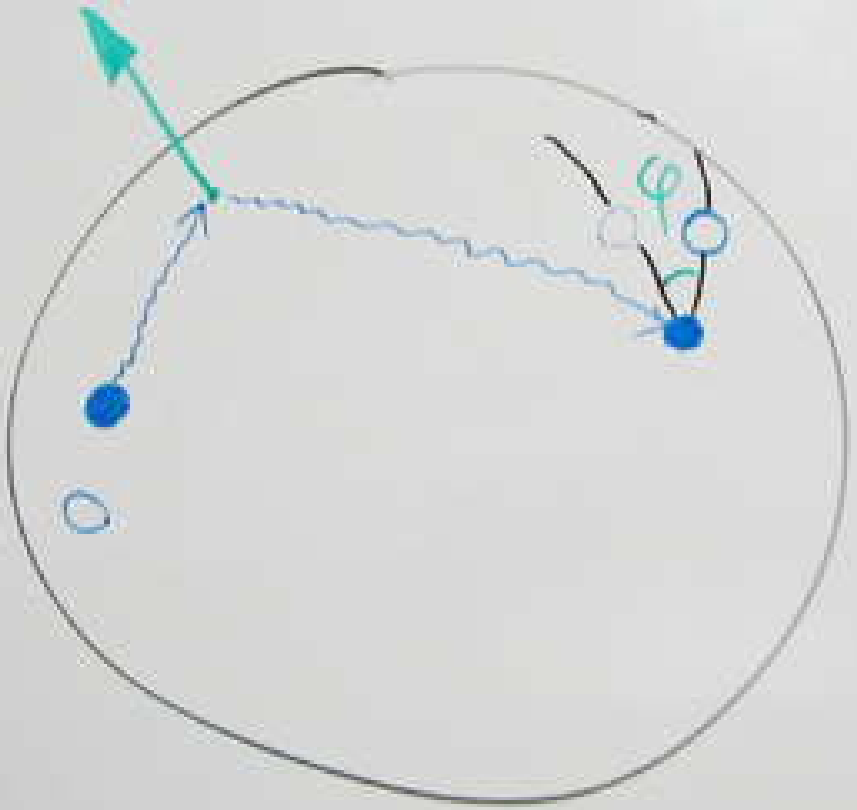
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

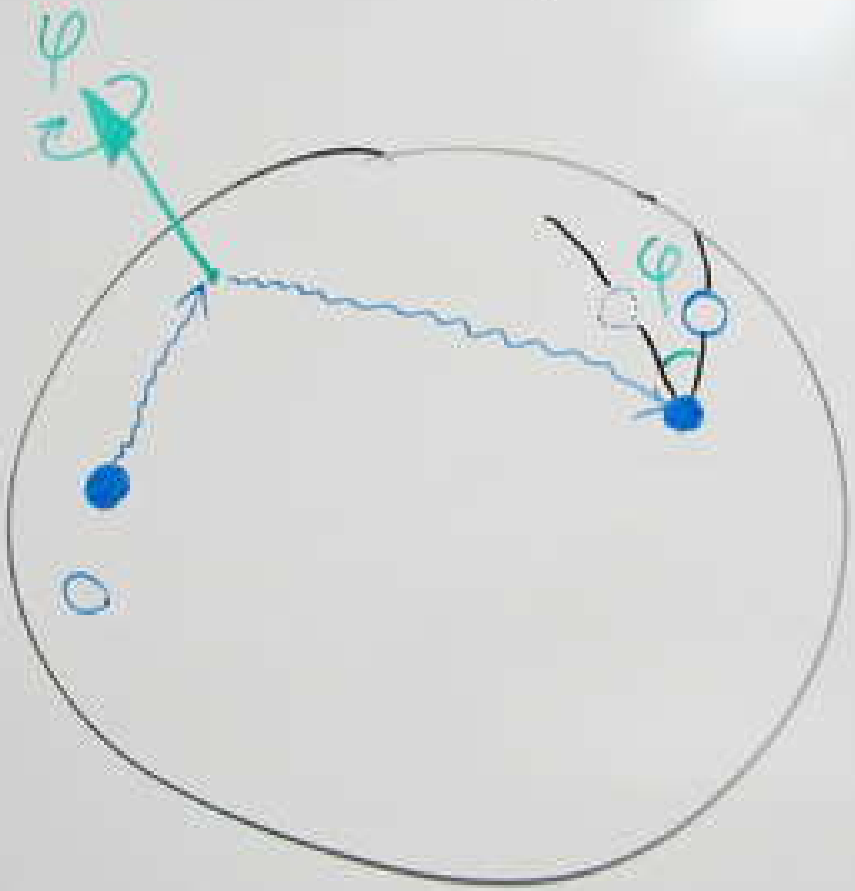
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

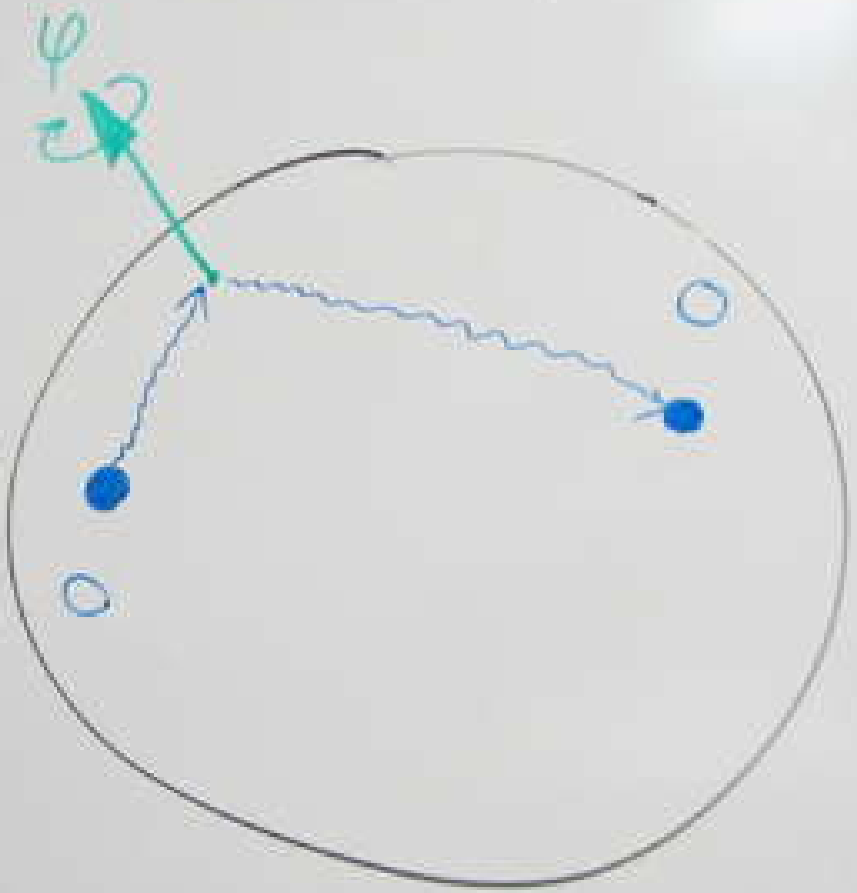
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

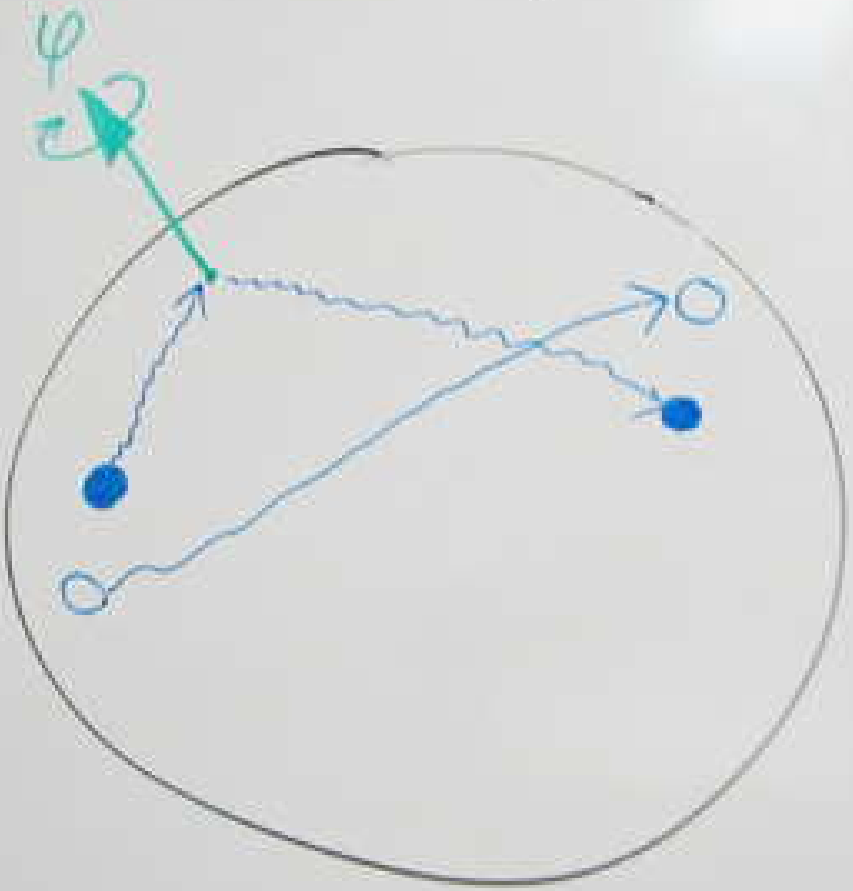
(fixed reference frame)



CASE I (∞, ∞)

PROOF 2

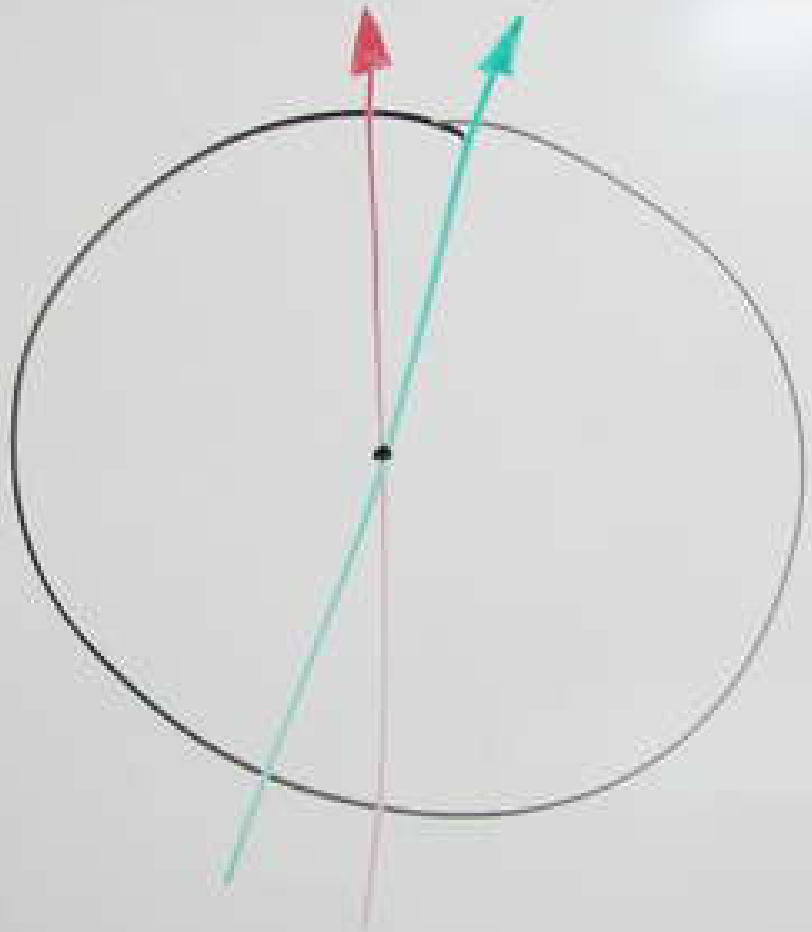
(fixed reference frame)



Case II (∞, n)

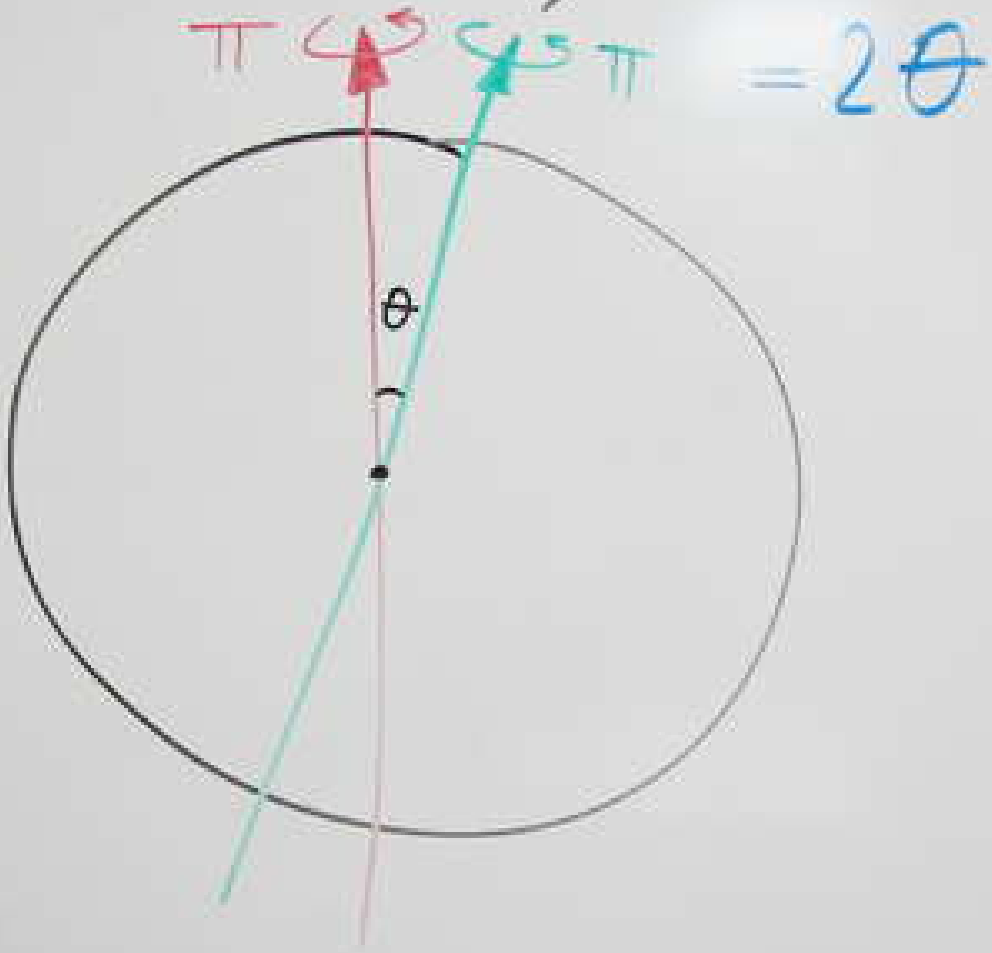
Case II (∞, n)

If n even:



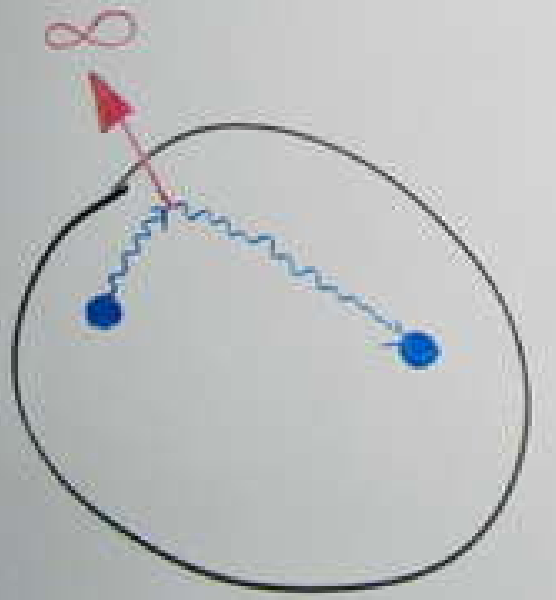
Case II (∞, n)

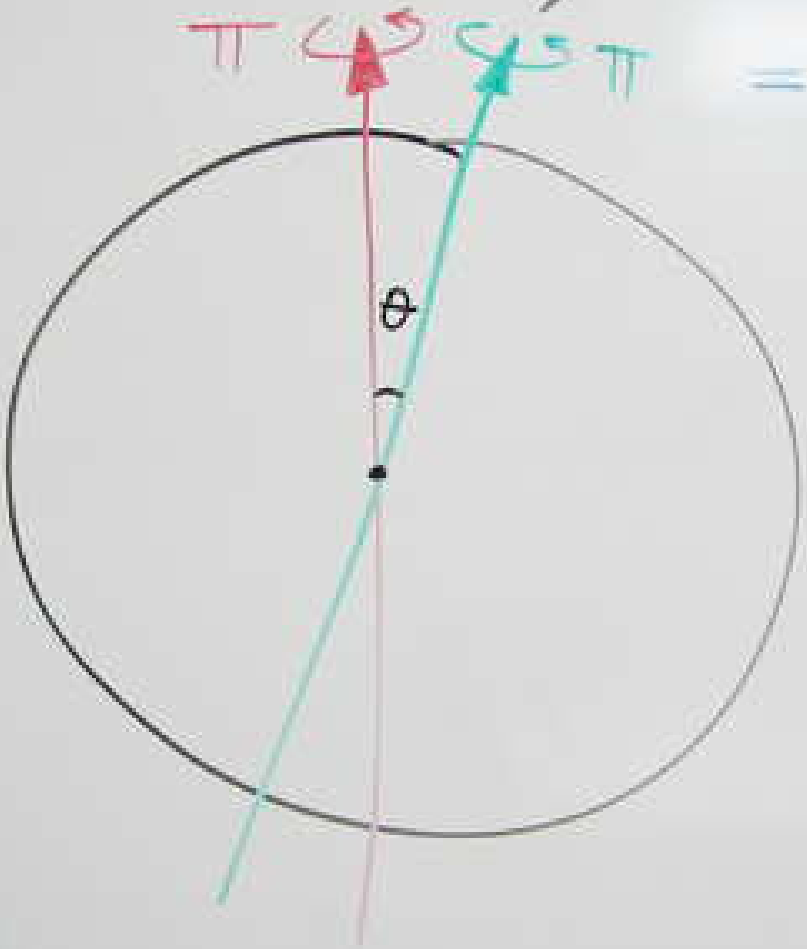
If n even:



Case II (∞, n)

If n even:





$= 2\theta$

Case II (∞, n)

For any n:

Case II (∞, n)

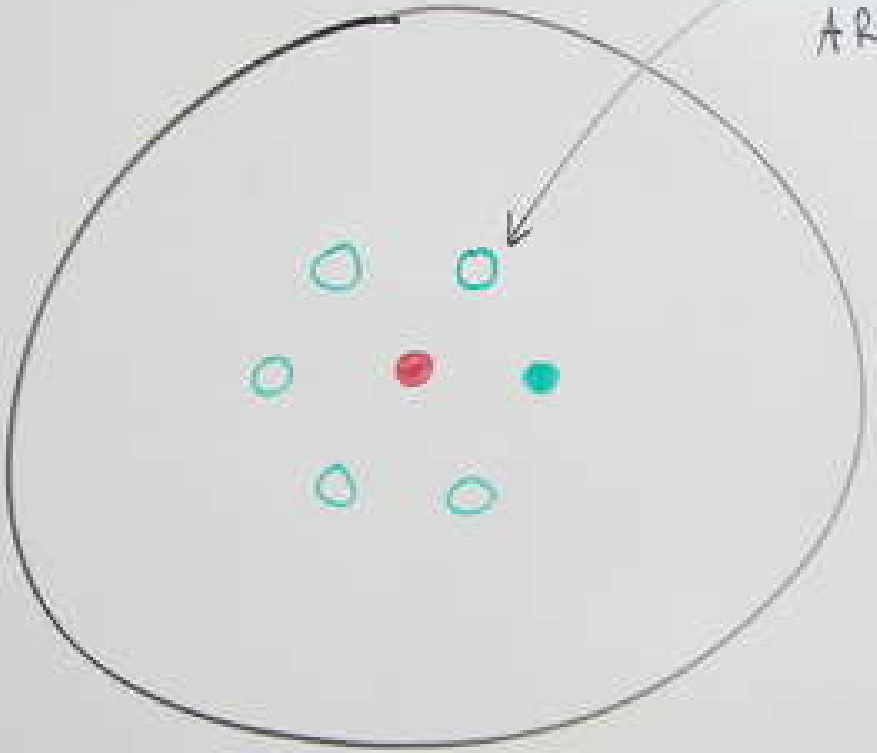
For any n:

Case II (∞, n)

For any n:

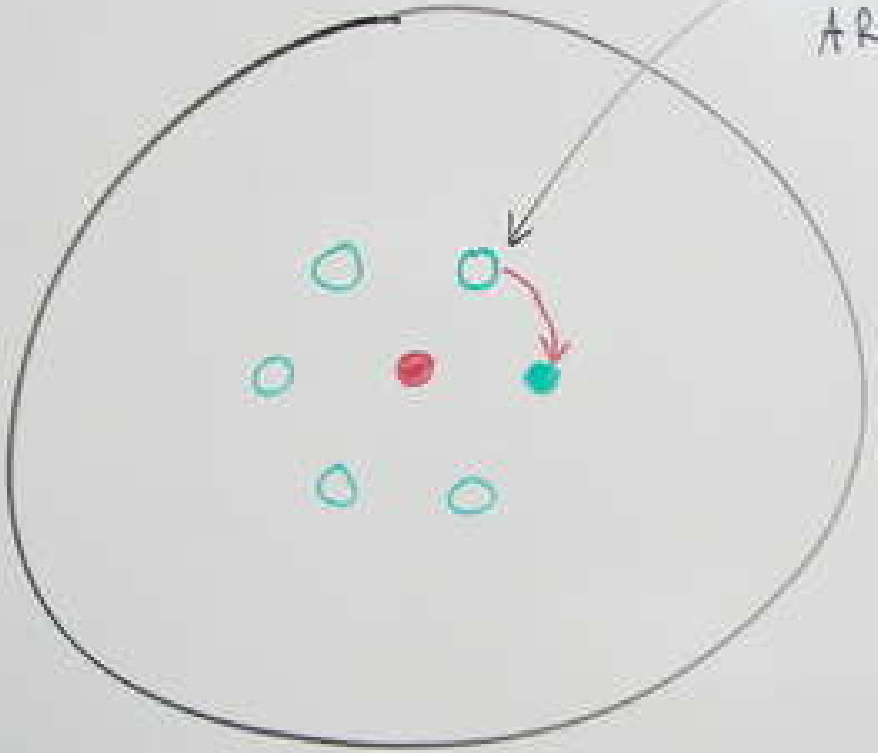
CASE II (∞, n)

For any n:



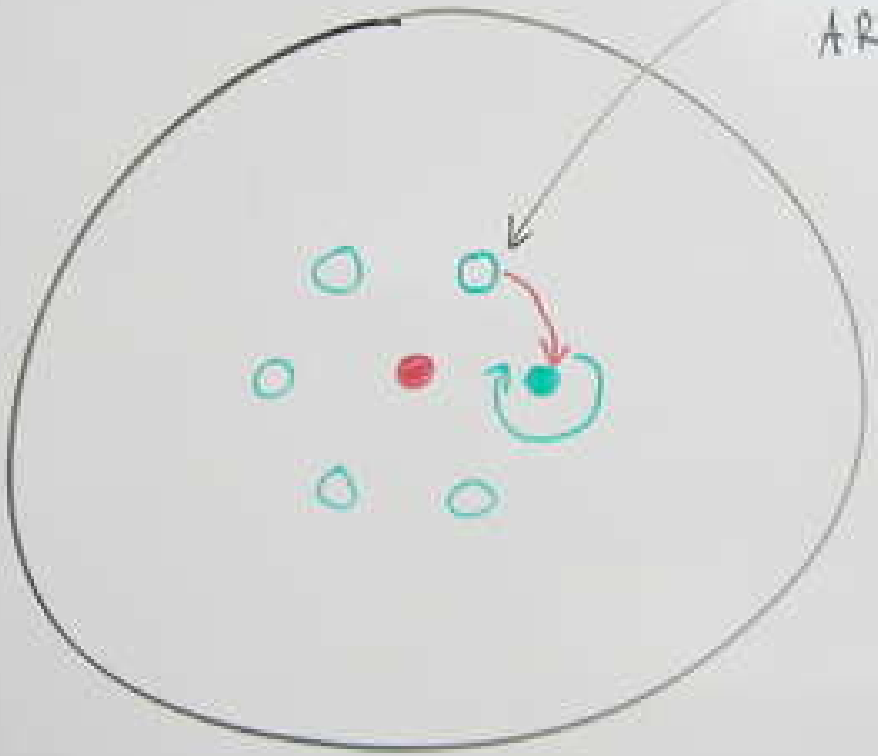
CASE II (∞, n)

For any n:



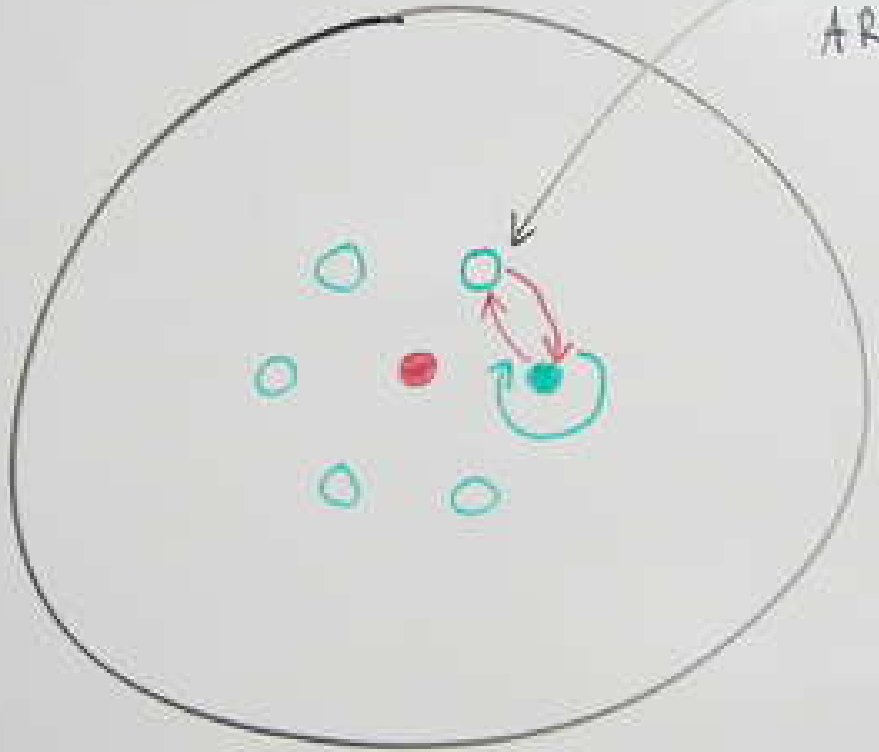
CASE II (∞, n)

For any n:



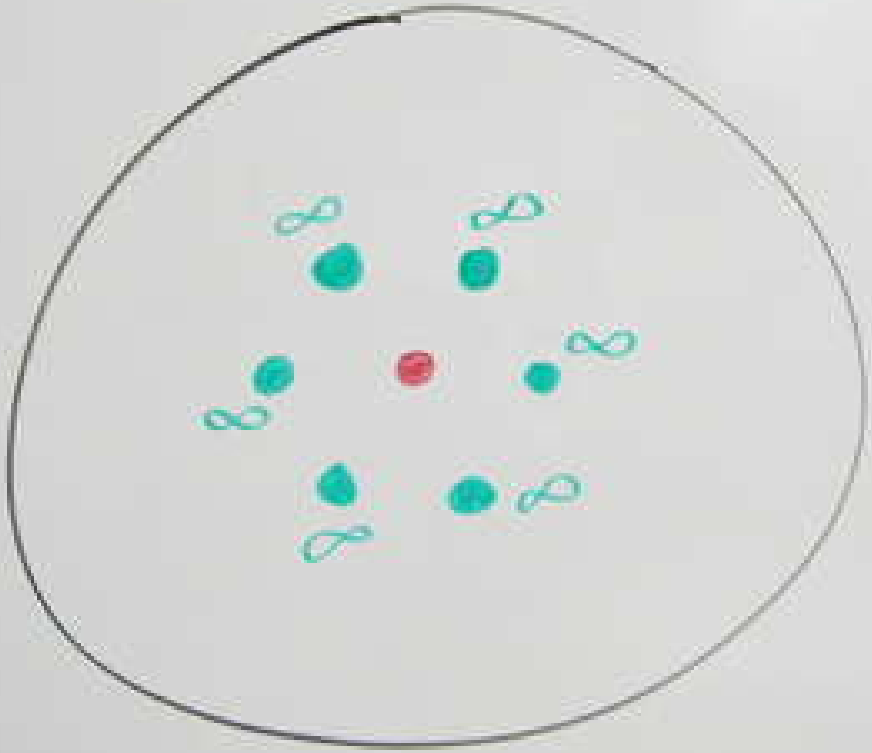
Case II (∞, n)

For any n:



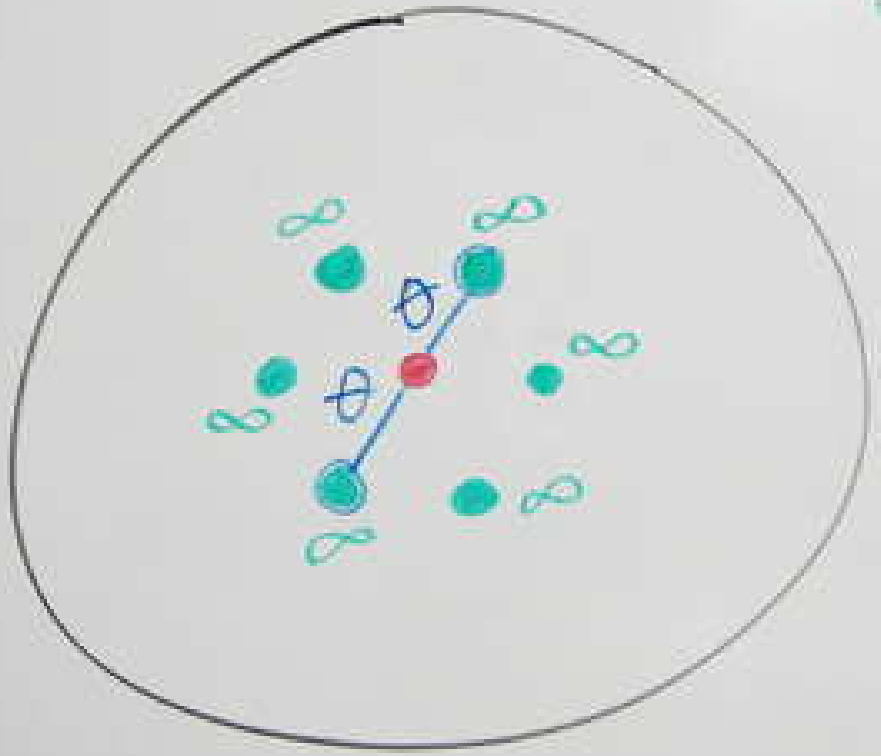
Case II (∞, n)

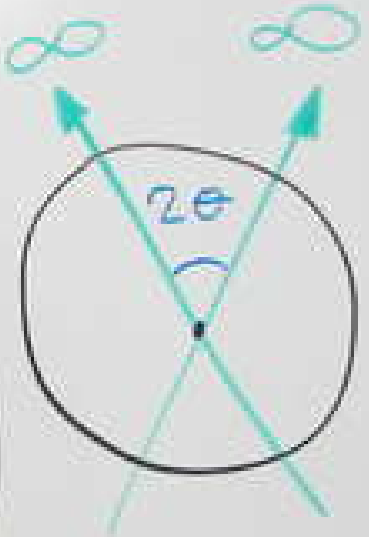
For any n:



Case II (∞, n)

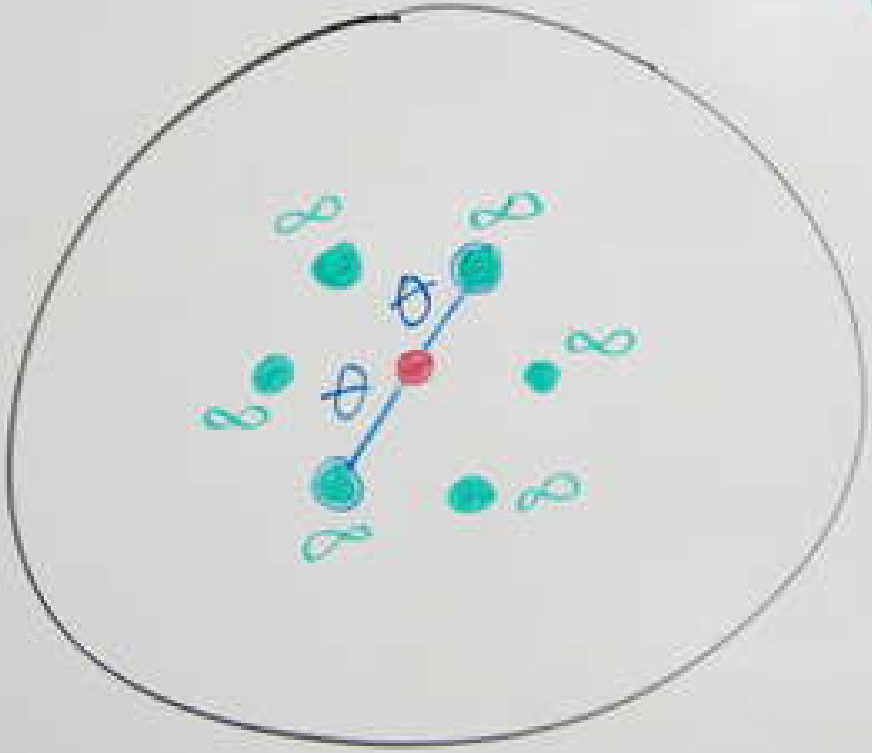
For any n:

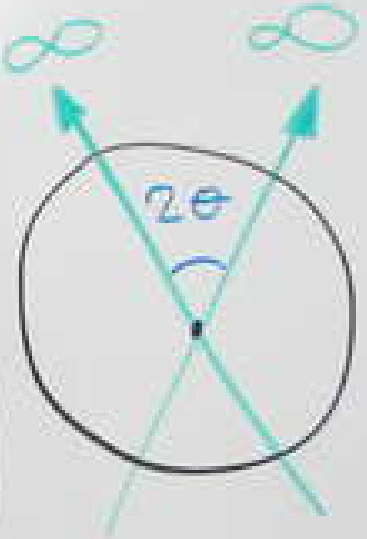




CASE II (∞, n)

For any n:

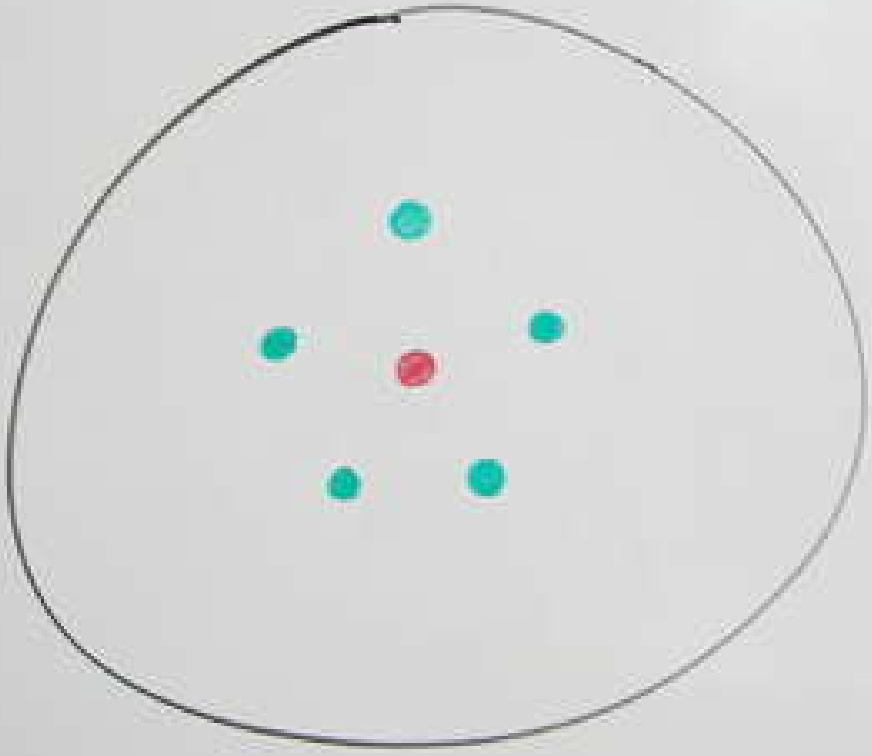




USE CASE I!

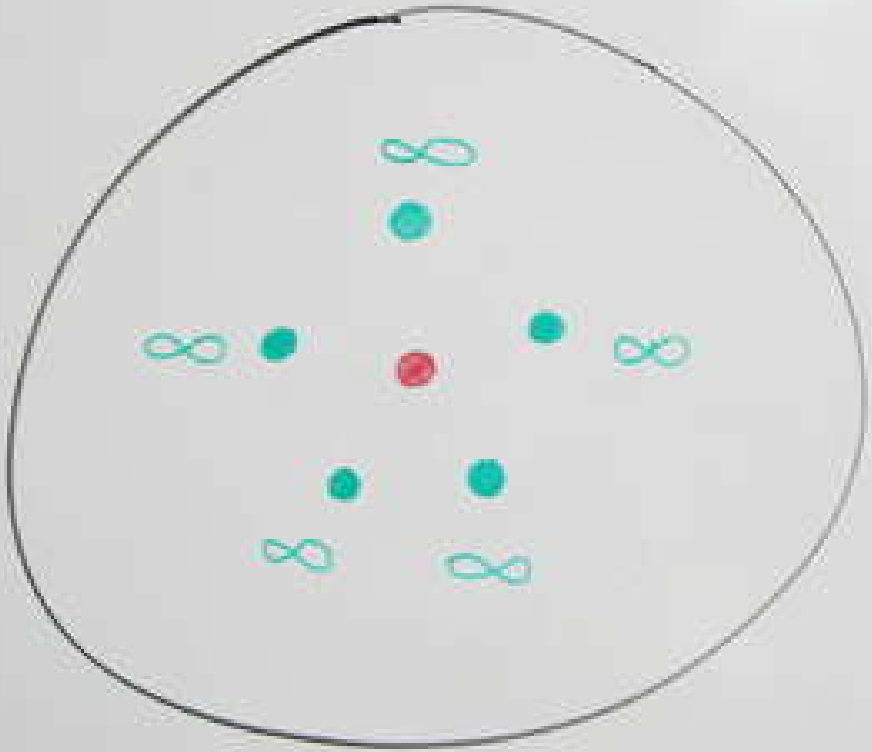
Case II (∞, n)

For any n:



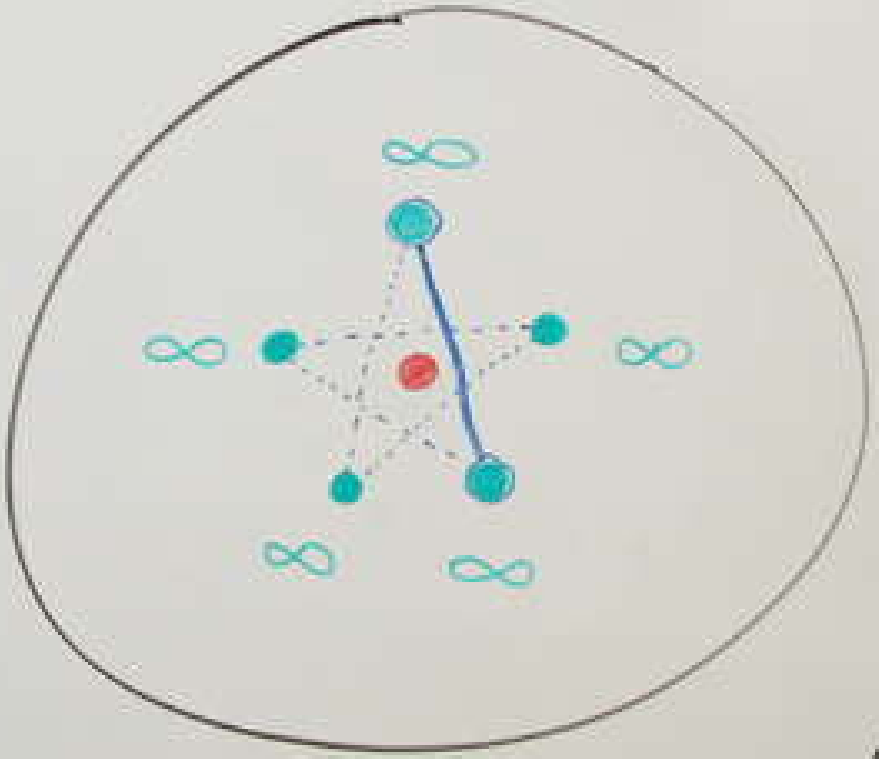
Case II (∞, n)

For any n:

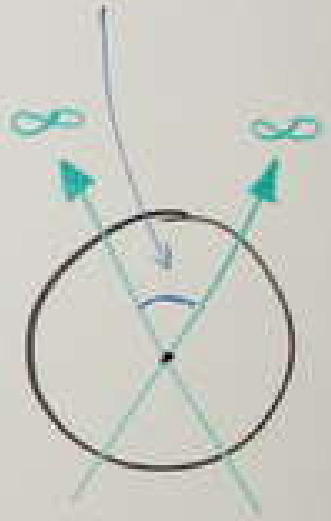


Case II (∞, n)

For any n:



Almost 2θ



USE CASE I!

Case III (n, m)

CASE III (n, m)

WHEN IS THE GROUP FINITE?

CASE III (n, m)

WHEN IS THE GROUP FINITE?

- C_k - CYCLIC

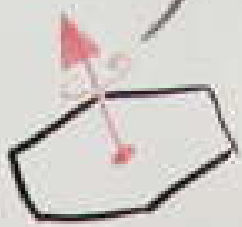
CASE III (n, m)

WHEN IS THE GROUP FINITE?

- C_k - CYCLIC

CASE III (n, m)

WHEN IS THE GROUP FINITE?

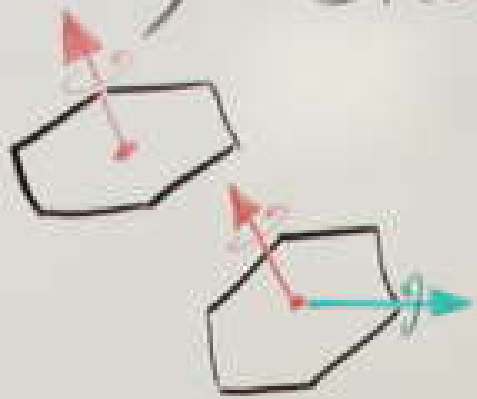


- C_k - CYCLIC
- D_k - DIHEDRAL

CASE III (n, m)

WHEN IS THE GROUP FINITE?

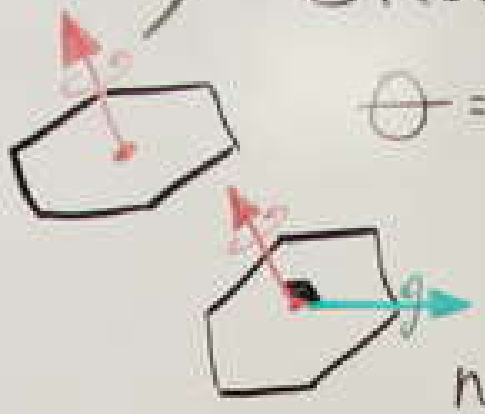
- C_k - CYCLIC
- D_k - DIHEDRAL



CASE III (n, m)

WHEN IS THE GROUP FINITE?

- C_k - CYCLIC
- D_k - DIHEDRAL



$\theta = 0$

$\theta = \frac{\pi}{2}$

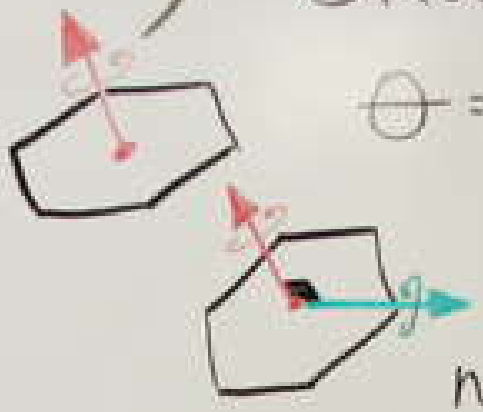
$n=2$ OR $m=2$

CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL



$\theta = 0$

$\theta = \frac{\pi}{2}$

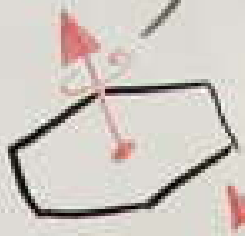
$n=2$ OR $m=2$

CASE III (n, m)

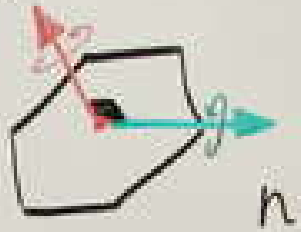
WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL

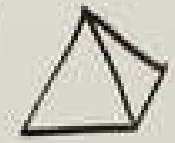


$\theta = 0$



$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$



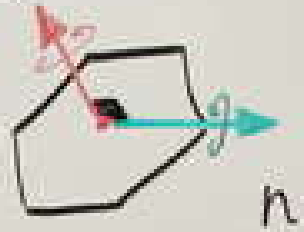
CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

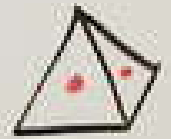
- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL

$\theta = 0$



$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$



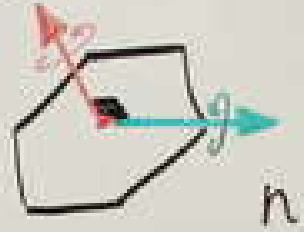
CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL

$\theta = 0$



$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

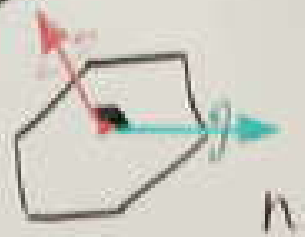
CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL

$\theta = 0$



$\theta = \frac{\pi}{2}$

$n = 2$ OR $m = 2$

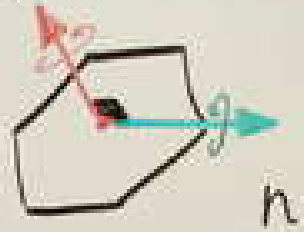
CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL

$\theta = 0$



$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

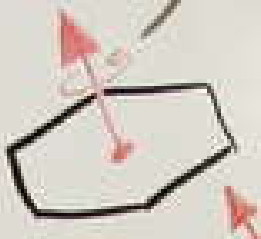
2 3 3

CASE III (n, m)

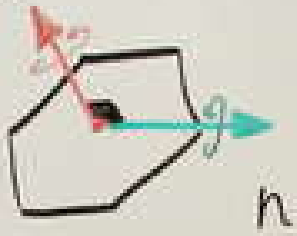
WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL
- O - OCTAHEDRAL



$\theta = 0$



$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

2 3 3

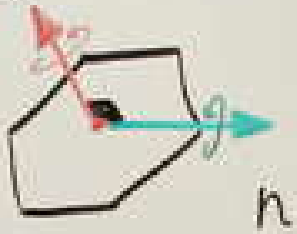
CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL
- O - OCTAHEDRAL

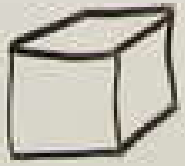
$\theta = 0$



$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

2 3 3



CASE III (n, m)

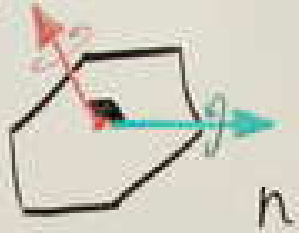
WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC

$\theta = 0$

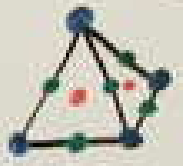
- D_k - DIHEDRAL



$\theta = \frac{\pi}{2}$

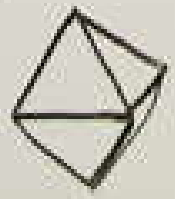
$n=2$ OR $m=2$

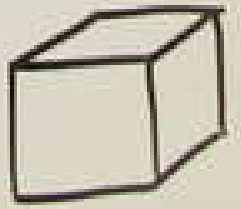
- T - TETRAHEDRAL



2 3 3

- O - OCTAHEDRAL



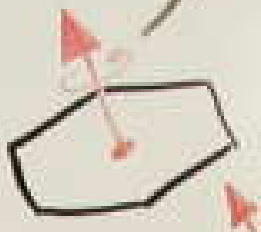


CASE III (n, m)

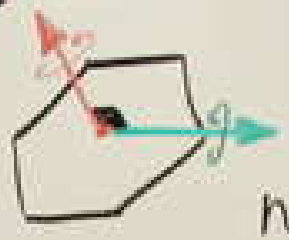
WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL
- O - OCTAHEDRAL



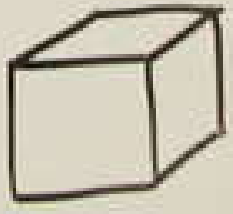
$\theta = 0$



$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

2 3 3

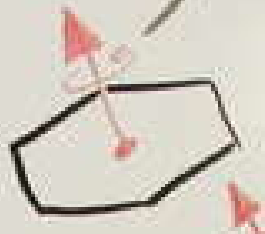


CASE III (n, m)

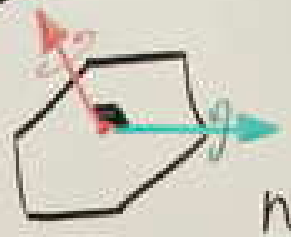
WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL
- O - OCTAHEDRAL



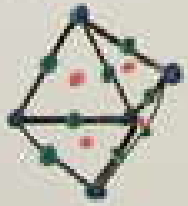
$\theta = 0$



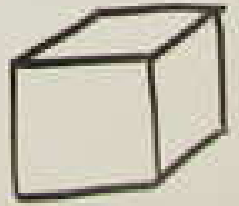
$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

2 3 3



2 4 3

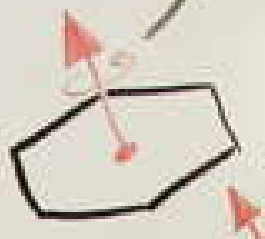


CASE III (n, m)

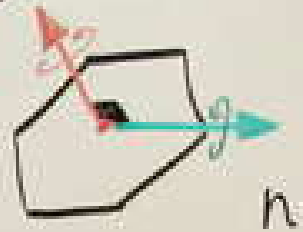
WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL
- O - OCTAHEDRAL



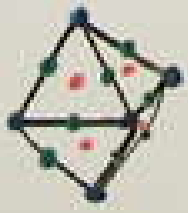
$\theta = 0$



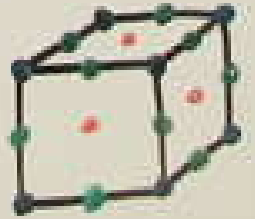
$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

2 3 3



2 4 3



CASE III (n, m)

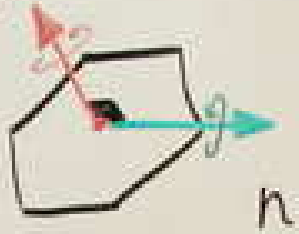
WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC

$\theta = 0$

- D_k - DIHEDRAL



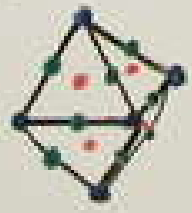
$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

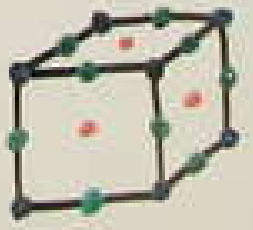
- T - TETRAHEDRAL

233

- O - OCTAHEDRAL



243



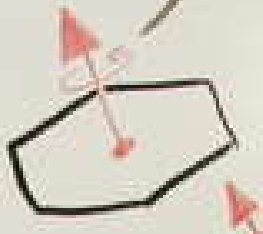
234

CASE III (n, m)

WHEN IS THE GROUP FINITE?

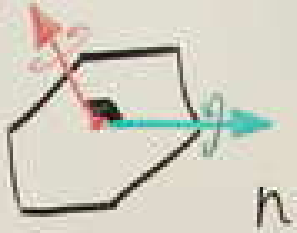
DEGENERATE

- C_k - CYCLIC



$\theta = 0$

- D_k - DIHEDRAL



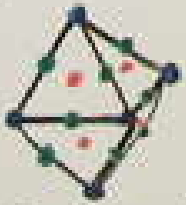
$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

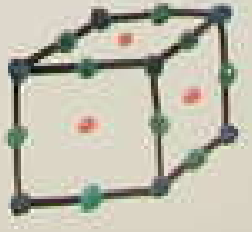
- T - TETRAHEDRAL

2 3 3

- O - OCTAHEDRAL



2 4 3



2 3 4

- I - ICOSAHEDRAL

CASE III (n, m)

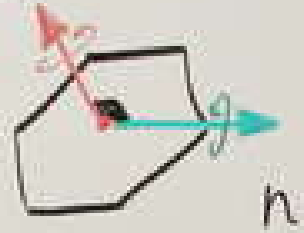
WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC

$\theta = 0$

- D_k - DIHEDRAL



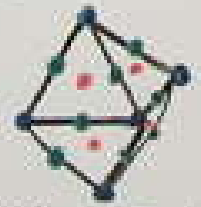
$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

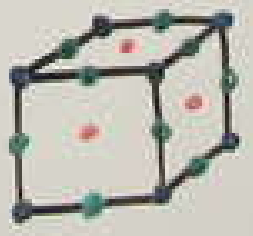
- T - TETRAHEDRAL

233

- O - OCTAHEDRAL



243



234

- I - ICOSAHEDRAL

CASE III (n, m)

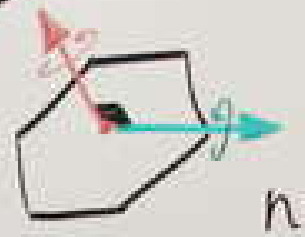
WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC

$\theta = 0$

- D_k - DIHEDRAL



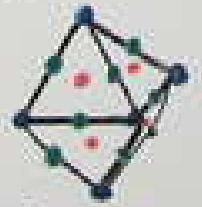
$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

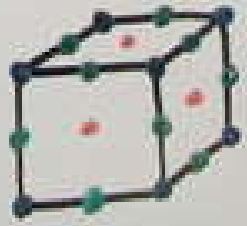
- T - TETRAHEDRAL

2 3 3

- O - OCTAHEDRAL

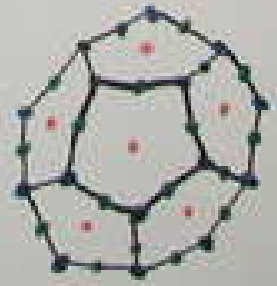


2 4 3



2 3 4

- I - ICOSAHEDRAL



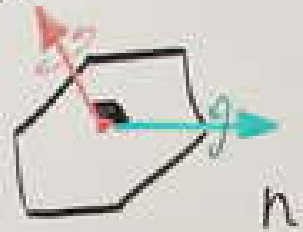
CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL
- O - OCTAHEDRAL
- I - ICOSAHEDRAL

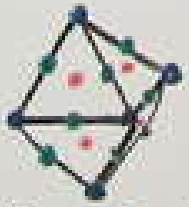
$\theta = 0$



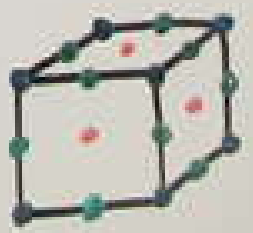
$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

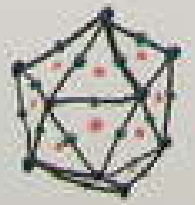
233



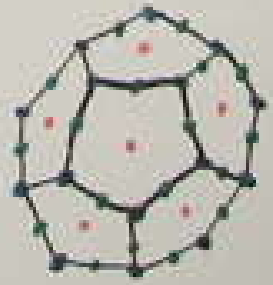
243



234



253



235

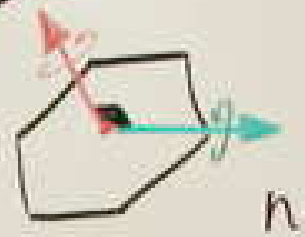
CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL

$\theta = 0$



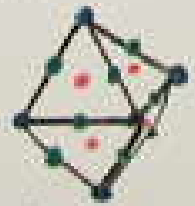
$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

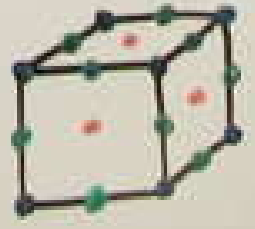
- 12 • T - TETRAHEDRAL

233

- 24 • O - OCTAHEDRAL

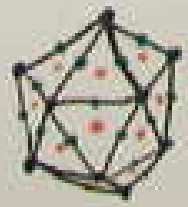


243

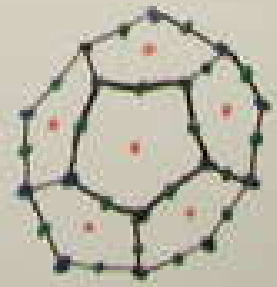


234

- 60 • I - ICOSAHEDRAL



253



235

CASE III (n, m)

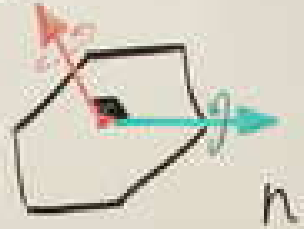
WHEN IS THE GROUP FINITE?

DEGENERATE

FINITE NUMBER

- C_k - CYCLIC
- D_k - DIHEDRAL
- T - TETRAHEDRAL
- O - OCTAHEDRAL
- I - ICOSAHEDRAL

$\theta = 0$



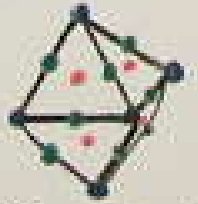
$\theta = \frac{\pi}{2}$

$n=2$ OR $m=2$

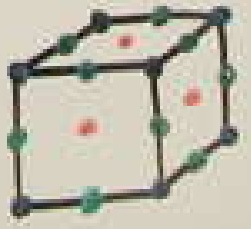
233

12

24

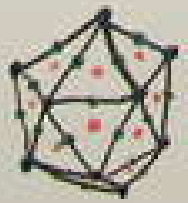


243

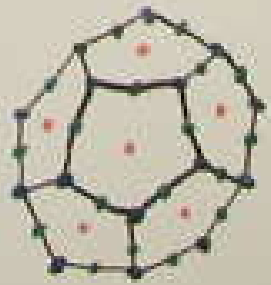


234

60



253



235

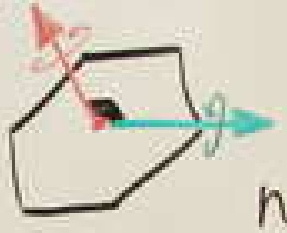
CASE III (n, m)

WHEN IS THE GROUP FINITE?

DEGENERATE

- C_k - CYCLIC
- D_k - DIHEDRAL

$\theta = 0$



$\theta = \frac{\pi}{2}$

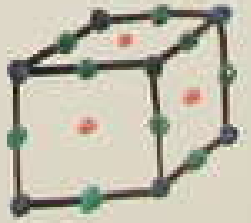
$n=2$ OR $m=2$

FINITE NUMBER

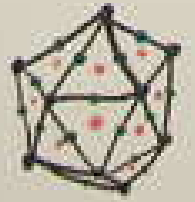
- T - TETRAHEDRAL (12)
- O - OCTAHEDRAL (24)
- I - ICOSAHEDRAL (60)

233

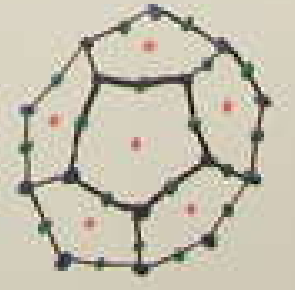
243



234



253



235

CASE III (n, m) WHEN IS THE GROUP

CASE III (n, m) WHEN IS THE GROUP INFINITE?

CASE III (n, m) WHEN IS THE GROUP INFINITE?

IN ALL OTHER CASES.

CASE III (n, m) WHEN IS THE GROUP INFINITE?

IN ALL OTHER CASES.

HOW DO WE GET AN IRRATIONAL ANGLE OUT OF TWO RATIONAL ANGLES?

CASE III (n, m) WHEN IS THE GROUP INFINITE?

IN ALL OTHER CASES.

HOW DO WE GET AN IRRATIONAL ANGLE OUT OF TWO RATIONAL ANGLES?

THEN WE CAN USE CASE II.

CASE III (n, m) WHEN IS TH GROUP INFINI

IN ALL OTHER CASES.

HOW DO WE GET AN IRATIONAL ANGLE OUT OF TWO RATIONAL ANGLES?

THEN WE CAN USE CASE II.

$\Gamma_{\alpha,0} = \{(0,0), (1, \frac{1}{2}+\alpha)\}, (2, \frac{1}{2}+2\alpha)$

Do not erase!

$y = x^2 + \alpha x + \beta$
$y - x^2 = \alpha x +$

CASE III (n, m) WHEN IS T
GROUP INFINI

IN ALL OTHER CASES.

HOW DO WE GET AN
IRRATIONAL ANGLE OUT OF
TWO RATIONAL ANGLES?

THEN

Do not erase!

$y = x^2 + \alpha x + \beta$

$y - x^2 = \alpha x + \beta$

$\sum_{x \in \mathbb{Z}_p} |x, \alpha x + \beta\rangle$

$|g\rangle$

CASE III (n, m) WHEN IS THE GROUP INFINITE?

IN ALL OTHER CASES.

HOW DO WE GET AN IRRATIONAL ANGLE OUT OF TWO RATIONAL ANGLES?

THEN WE CAN USE CASE II.

$P_{\alpha,0} = \{(0,0), (1, \tan(\alpha)), (2, \tan(2\alpha)) \ldots\}$

Do not erase!

CASE III (n, m) WHEN IS THE GROUP INFINITE?

IN ALL OTHER CASES.

HOW DO WE GET AN IRRATIONAL ANGLE OUT OF TWO RATIONAL ANGLES?

THEN WE CAN USE CASE II

Do not erase!

CASE III (n, ∞) WHEN IS THE GROUP INFINITE?

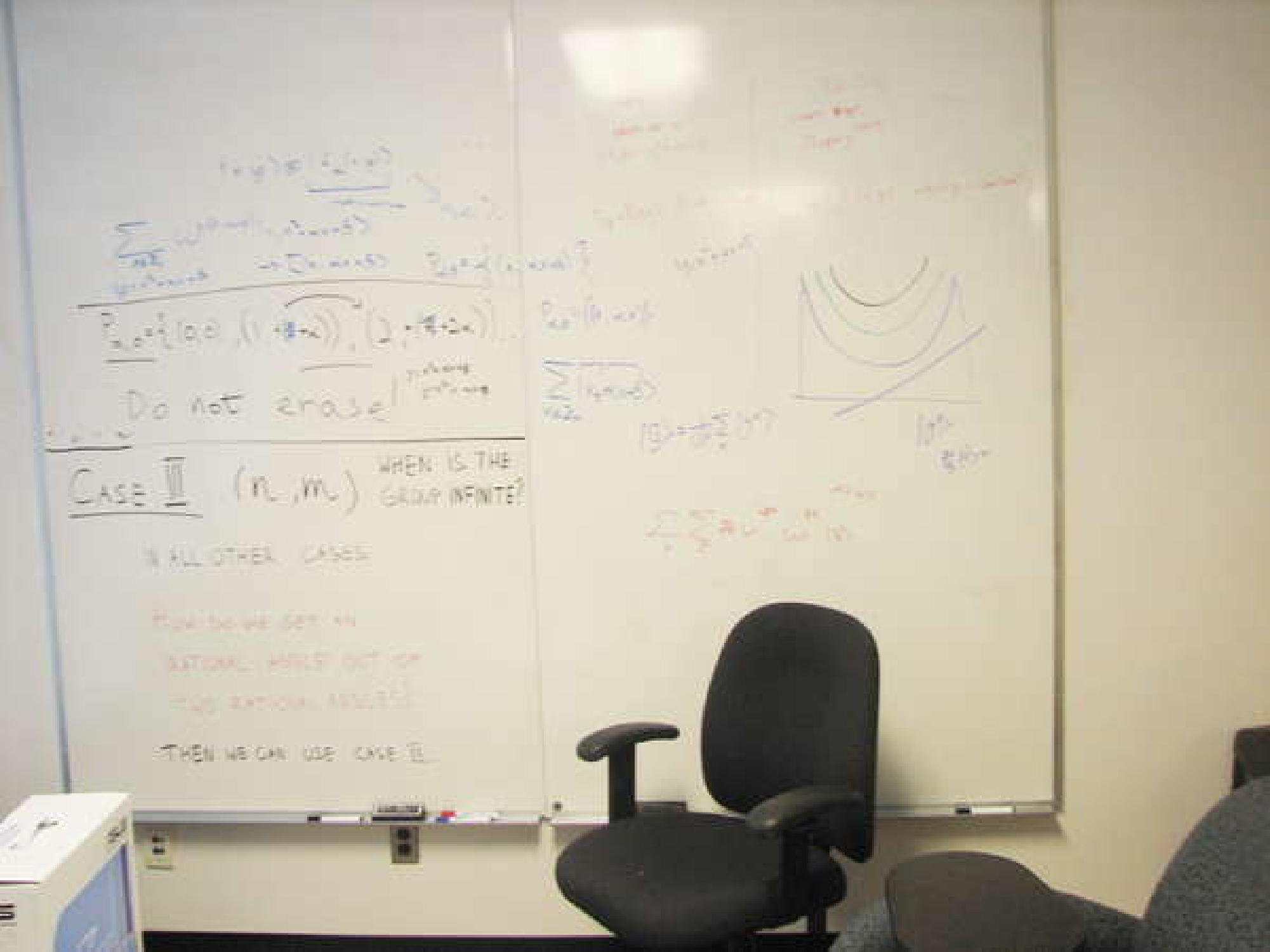
IN ALL OTHER CASES

HOW DO WE GET AN

NATURAL MAP OUT OF

TWO NATURAL MAPS?

THEN WE CAN USE CASE II.


Do not erase!
CASE II
WHEN IS THE GROUP INFINITE?
IN ALL OTHER CASES
THEN WE CAN USE CASE II


CLEAN
YOUR
DISHES
RIGHT NOW


CLEAN
YOUR
DISHES
RIGHT NOW


CLEAN
YOUR
DISHES
RIGHT NOW


NOW


9:04
Lipton
Lipton


Lipton
Caramel Twist
Lipton


5:05
Lipton


THE END


THE END


THE END


THE END


THE END